Kakeya Sets

Jonathan Hickman

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The Kakeya Problem

Definition
A Kakeya set $K \subset \mathbb{R}^n$ is a compact subset which contains a unit line segment in every direction.
Examples of Kakeya subsets of the plane:

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- Area = $\frac{\pi}{4}$
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- Area = $\frac{\pi}{8}$
The Kakeya Problem

Question

What is the smallest possible area of a Kakeya set in the plane?
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Answer (Besicovitch, 1919)
There exists a Kakeya $K \subseteq \mathbb{R}^2$ set of plane measure zero.
Construction of a Besicovitch set

**Figure:** Begin with an equilateral triangle of height 1. Partition the base into $2^k$ equal intervals and use the intervals to form $2^k$ triangles.
Construction of a Besicovitch set

**Figure:** Slide the small triangles horizontally and bunch them together so they have large overlap. Note the resulting figure will still contain a unit line in a range of directions covering $2\pi/3$ radians.
Construction of a Besicovitch set

Figure: Continue to slide the triangles horizontally.
Construction of a Besicovitch set

**Figure:** The final configuration of triangles has much smaller area than the original figure. We call it a **Perron tree**.
Construction of a Besicovitch set

Figure: The final configuration of triangles has much smaller area than the original figure. We call it a **Perron tree**. Moreover, by choosing $k$ sufficiently large, the area of the Perron tree can be made arbitrarily small.
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Kakeya sets over Finite Fields

Besicovitch sets

There have been numerous proofs of existence of Kakeya sets of measure zero. Notably:

▶ Kahane (1969) - by joining points of two parallel Cantor-like sets.
▶ There exists a set $K$ of measure zero containing a full line in every direction (see Falconer's book).
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- There exists a set $K$ of measure zero containing a full line in every direction (see Falconer’s book).
Although the Kakeya sets seem like a mere curiosity, they have been found to have numerous applications in various fields:

- Harmonic analysis (Fefferman)
- Study of solutions to the wave equation (Wolff)
- Additive combinatorics (Bourgain)
- Analytic number theory (Bourgain)
- Cryptography (Bourgain)
- Random number generation in computer science (Dvir, Wigderson)

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Here I’ll briefly describe Fefferman’s classical (and ingenious!) application of Kakeya sets to the **ball multiplier problem from Fourier analysis.**
Connections with Harmonic Analysis

Theorem (M. Riesz, 1928)

Let $Q := [-1, 1]^2$ and

$$Sf = (\hat{f} \chi_Q) \ast f \in \mathcal{S}(\mathbb{R}^2).$$

Then $S$ is a bounded operator on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$. 
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Then \( S \) is a bounded operator on \( L^p(\mathbb{R}^2) \) for \( 1 < p < \infty \).

Theorem (C. Fefferman, 1971)
Let \( B := \{ x \in \mathbb{R}^2 : |x| \leq 1 \} \) and
\[
Tf = (\hat{f} \chi_B)^\sim \quad f \in \mathcal{S}(\mathbb{R}^2).
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Then \( T \) is a bounded operator on \( L^p(\mathbb{R}^2) \) if and only if \( p = 2 \).
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At the heart of Fefferman’s proof lies (a variant of) the Perron tree described above.
Fefferman takes $f$ to be the sum of characteristic functions of certain disjoint long thin tubes, multiplied by certain phase factors.

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Thus, $T$ does not preserve the qualitative properties of the distribution of $f$ and fails to be bounded on $L^p$ for $p \neq 2$. 
Introducing discretisation: replaced line segments in $K$ with, say, a large collection of $1 \times \delta$ tubes for $0 < \delta \ll 1$. 
Reviewing Fefferman’s proof

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- Important to understand the optimal compression / pile-up for these tubes.
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- Introduced discretisation: replaced line segments in $K$ with, say, a large collection of $1 \times \delta$ tubes for $0 < \delta \ll 1$.
- Important to understand the optimal compression / pile-up for these tubes.
- By arranging the tubes in a Perron-tree we achieved a high level of tube pile-up - led to bad behaviour.
Kakeya Conjecture

Given a Kakeya set $K$, the Minkowski dimension of $K$ tells us how large the resulting set will be if we fatten the lines in $K$ to $\delta$ tubes.
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**Definition**

A subset $E \subseteq \mathbb{R}^n$ has Minkowski dimension at least $\alpha$ if for any $0 < \epsilon \leq 1$ there exists a constant $C_\epsilon$ such that

$$|E_\delta| \geq C_\epsilon \delta^{n-\alpha+\epsilon}$$
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**Conjecture**

*Any Kakeya set $K \subseteq \mathbb{R}^n$ has Minkowski dimension $n$.***
The Kakeya conjecture essentially tells us any pile-up of tubes cannot be much worse than what we have already experienced.

This is related to many problems in PDE and Fourier analysis...
Kakeya Conjecture

Local Smoothing
\[\Downarrow\]
Maximal Bochner-Riesz
\[\Downarrow\]
Spherical Bochner-Riesz
\[\Downarrow\]
Spherical Restriction
\[\Downarrow\]
Parabolic Restriction \[\iff\] Parabolic Bochner-Riesz
\[\Downarrow\]
Maximal Kakeya
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- Dimension 2 case is known (Davies 1971, Cordoba 1977).
- Partial results are known in all dimensions, but obtaining sharp results seems very difficult.
- Wolff proposed a finite field analogue of the Kakeya conjecture to act as a toy model.
Definition
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$$\{y + t\omega : t \in \mathbb{F}\}$$

lies in $K$. 

Note $\omega_1, \omega_2 \in \mathbb{F}^n \setminus \{0\}$ define the same direction if $\omega_1 = t\omega_2$ for some $t \in \mathbb{F}$. Therefore there are $|\mathbb{F}|^n - 1$ distinct directions in $\mathbb{F}^n$. 

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Note $\omega_1, \omega_2 \in F^n \setminus \{0\}$ define the same direction if $\omega_1 = t\omega_2$ for some $t \in F$. Therefore there are
$$\frac{|F|^n - 1}{|F| - 1} \sim |F|^{n-1}$$
distinct directions in $F^n$. 
Conjecture (Finite field Kakeya conjecture)

If $K \subseteq \mathbb{F}^n$ is a Kakeya set, then

$$|K| \geq c_n|\mathbb{F}|^n$$

where $0 < c_n$ is a constant depending only on $n$. 
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- There are \( \sim |\mathbb{F}|^{n-1} \) lines in \( K \) pointing in different directions;
- Each line contains \( |\mathbb{F}| \) elements;
- The conjecture states union of these lines has \( \sim |\mathbb{F}|^n \) elements - therefore the lines are essentially disjoint.

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Kakeya Sets
Initially very little progress was made on this conjecture (despite the attention of Wolff, Mockenhaupt, Rogers, Tao, et al.).
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It could easily fit in a first year linear algebra course!
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- Use the structure of a Kakeya set to show any polynomial of low degree which vanishes on $K$ must be the zero polynomial.
Kakeya sets over Finite Fields

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- Conclude \( K \) is not small.
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- Use the structure of a Kakeya set to show any polynomial of low degree which vanishes on $K$ must be the zero polynomial.
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Lemma

If $E \subset \mathbb{F}^n$ with $|E| < \binom{n+d}{n}$, then there exists a non-zero polynomial of degree at most $d$ which vanishes on $E$. 
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Proof.
The space of polynomials of degree at most $d$ on $\mathbb{F}^n$ is $\binom{n+d}{n}$-dimensional. The linear map

$$\text{eval} : \mathbb{F}^{|E|} \rightarrow \mathbb{F}^{|E|}$$
$$\text{eval} : P \mapsto (P(x))_{x \in E}$$

therefore has non-trivial kernel.
Kakeya sets over Finite Fields

- Show if $E \subseteq \mathbb{F}^n$ is small, then there exists a non-zero polynomial of low degree which vanishes on $E$. ✓
- Use the structure of a Kakeya set to show any polynomial of low degree which vanishes on $K$ must be the zero polynomial.
Proposition

If $P : \mathbb{F}^n \to \mathbb{F}$ is a polynomial of degree at most $|\mathbb{F}| - 1$ which vanishes on $K$, then $P$ is identically zero.
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- Assume \( P \neq 0 \). Then \( P = \sum_{i=0}^{d} P_i \) where each \( P_i \) is homogeneous of degree \( i \) and \( P_d \neq 0 \). Since \( P \) vanishes on \( K \), the polynomial is non-constant and \( d > 0 \).
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- Given a direction $\omega \in \mathbb{F}^n \setminus \{0\}$, there exists $y \in \mathbb{F}^n$ such that $P$ vanishes on the line $\{y + t\omega : t \in \mathbb{F}\}$.
- Hence $Q(t) := P(y + t\omega)$ is a univariate polynomial of degree less than $|\mathbb{F}|$ which vanishes for all $t \in \mathbb{F}$. 

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- Given a direction $\omega \in \mathbb{F}^n \setminus \{0\}$, there exists $y \in \mathbb{F}^n$ such that $P$ vanishes on the line $\{y + t\omega : t \in \mathbb{F}\}$.
- Hence $Q(t) := P(y + t\omega)$ is a univariate polynomial of degree less than $|\mathbb{F}|$ which vanishes for all $t \in \mathbb{F}$.
- $Q$ is identically zero and, in particular the $t^d$ coefficient, which is $P_d(\omega)$, is zero.
- Therefore $P_d(x) = 0$ for all $x \in \mathbb{F}^n$. Since the degree of $P_d$ is less than $|\mathbb{F}|$, one concludes $P_d$ is identically zero, a contradiction.
Corollary

Any Kakeya set $K \subseteq \mathbb{F}^n$ has cardinality

$$|K| \geq \binom{n + |\mathbb{F}| - 1}{n} \geq \frac{1}{n!}|\mathbb{F}|^n$$
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- If $E \subset \mathbb{F}^n$ with $|E| < \binom{n+d}{n}$, then there exists a non-zero polynomial of degree at most $d$ which vanishes on $E$.
- If $P : \mathbb{F}^n \to \mathbb{F}$ is a polynomial of degree at most $|\mathbb{F}| - 1$ which vanishes on a Kakeya set $K$, then $P$ is identically zero.
Impact

- The (continuous) Kakeya conjecture remains open.
- Multilinear-Kakeya
- Numerous combinatorial problems amenable to polynomial method: solution to the Joints problem and Erdös distance conjecture.