1 Introduction

Recall, for an ensemble $(\Sigma, \{M_j\}_{j=1}^k, \{\pi_j\}_{j=1}^k)$ with associated form

$$T(f_1, \ldots, f_k) := \int_{\Sigma} \prod_{j=1}^k f_j \circ \pi_j(x) a(x) dx,$$

understanding the $L^p$-boundedness of $T$ (at least up to endpoints) can be reduced to studying certain “isoperimetric inequalities”. In particular, we wish to establish for which exponents $p = (p_1, \ldots, p_k) \in [1, \infty]^k$ the estimate

$$|U| \approx p_k \prod_{j=1}^k |\pi_j(U)|^{1/p_j}$$

holds for all open sets $U \subseteq \Sigma$ in a sufficiently small neighbourhood of the origin. We have already seen that (1) holds for non-trivial values of $p$ only if the family $X_\pi$ of associated vector fields satisfies the Hörmander condition. A deeper fact is that the converse of this statement is true. This converse may be expressed in a very precise fashion which relates the range of Lebesgue exponents for which (1) holds to the degrees of the Hörmander-tuples. This result will be discussed in some detail in the following notes.

To begin we give a heuristic argument which provides some necessary conditions on $p$ for (1) to hold, stated in terms of the degrees of the Hörmander-tuples. This line of reasoning will lead to a conjecture on the sharp range of $p$ for which $T$ is strong-type.

Before we proceed it is useful to once again reformulate the problem. Suppose $(p_1, \ldots, p_k)$ satisfy

$$\sum_{j=1}^k \frac{1}{p_j} > 1$$

(so we are in the non-trivial regime) and $U \subseteq \Sigma$ is an open set in a neighbourhood of 0. Define the quantities

$$\alpha_j := \frac{|U|}{|\pi_j(U)|} \quad \text{for } 1 \leq j \leq k.$$

Note, by the co-area formula $\alpha_j$ is the average length of the $\pi_j$-fibres through $U$. By some simple algebraic manipulation, (1) is equivalent to the estimate

$$|U| \approx p \prod_{j=1}^k \alpha_j^{b_{j}(p)},$$

(2)
where \( b(x) = (b_1(x), \ldots, b_k(x)) \) is defined by

\[
b_j(x) := \frac{1/x_j}{\sum_{i=1}^{k} 1/x_i - 1} \quad \text{for } x = (x_1, \ldots, x_k) \in (0, \infty)^k.
\]

We therefore wish to deduce the values of \( p \) for which (2) holds.

### 2 \( k \)-parameter Carnot-Carathéodory Balls

Throughout our study a certain family of “balls”, defined in terms of the associated vector fields, play a central rôle.

**Definition 1 (Carnot-Carathéodory Balls).** Let \( x \in \Sigma \) be close to 0 and \( \delta = (\delta_1, \ldots, \delta_k) \) where \( 0 < \delta_j \lesssim 1 \) are sufficiently small for \( 1 \leq j \leq k \). Define the \( k \)-parameter Carnot-Carathéodory ball \( B(x; \delta) \) to be the closure of the set of points

\[
e^{t_N \delta_j X_{jN}} \circ e^{t_{N-1} \delta_{j_{N-1}} X_{j_{N-1}}} \circ \cdots \circ e^{t_1 \delta_{j_1} X_{j_1}}(x),
\]

where \( N \in \mathbb{N}_0 \), \( 1 \leq j_i \leq k \) and \( \sum_{i=1}^{N} |t_i| \leq 1 \).

Thus, \( B(x; \delta) \) represents the set of points which can be reached by starting at \( x \) and flowing \( t_1 \) in the \( \delta_{j_1} X_{j_1} \) direction, then \( t_2 \) in the \( \delta_{j_2} X_{j_2} \) direction and so on.

These balls are useful sets on which to test inequalities of the form (2). Indeed, if we temporarily cast our minds back to the situation where the Hörmander condition fails we see such balls have (essentially) already appeared in our analysis. In this situation, by the quantitative Frobenius theorem, if one flows from 0 for a short period of time in the direction of any of the vector fields \( X_j \), then one remains close to some submanifold \( S \) of positive co-dimension. In particular, the Carnot-Carathéodory ball \( B(0; \delta) \) with \( \delta_1 = \cdots = \delta_k \) should resemble a thin neighbourhood of \( S \). In the proof of Theorem 8, we saw the operator \( T \) fails to be \( L^p \)-improving precisely by testing the (isoperimetric formulation of the) inequality on such a neighbourhood; namely,

\[
U := \{ x \in \Sigma : |x| \lesssim \delta \text{ and dist}(x, S) \lesssim \delta^N \}.
\]

Thus, we can think of the set \( U \) from the proof of Theorem 8 as roughly an approximation to \( B(0; \delta) \).

Returning to the situation where the vector fields do obey the Hörmander condition, by testing (2) on a Carnot-Carathéodory balls of small radii we hope to deduce necessary conditions on the Lebesgue exponents. Unfortunately the study of such balls can be rather complicated and obtaining good estimates for quantities such as \( |B(0; \delta)| \) can be arduous. For this reason it is expedient to begin by arguing purely heuristically. For instance, from the definition of the balls one expects the average length of the \( \pi_j \)-fibre to be roughly \( \delta_j \) which leads to the prediction

\[
\alpha_j := \frac{|B(0; \delta)|}{|\pi_j(B(0; \delta))|} \sim \delta_j.
\]

Indeed, consider the ball \( B_{1/2}(0; \delta) \) given by the collection of points which can be reached by starting at 0 and flowing for time at most \( 1/2 \) (rather than 1) in the directions of the \( \delta_j X_j \); that is, the closure of the set of points of the
form (3) (with $x := 0$) where $N \in \mathbb{N}_0$ and $1 \leq j_i \leq k$ as before, but now 
$\sum_{i=1}^{N} |t_i| \leq 1/2$. For any point in $B_{1/2}(0; \delta)$ one may still flow for time $1/2$
 in the $\delta_j X_j$ direction and remain in $B(0; \delta)$ and so the $\pi_j$-fibre through such a point
is of length similar to $\delta_j$. Since one expects $B_{1/2}(0; \delta)$ to be comparable
in volume to $B(0; \delta)$, our heuristic (4) follows.¹

Thus we may use these bounds to estimate the right-hand side of the re-
formulated inequality (2) when tested against the set $B(0; \delta)$. Importantly, we
can also get a good idea of the volume of these balls by arguing heuristically.
It turns out that whereas $\alpha_j \sim \delta_j$, the volume $|B(0; \delta)|$ of the ball depends
on the degrees of certain Hörmander-tuples. This can be derived heuristically
by considering the Baker-Campbell-Hausdorff formula, described below. This
formula allows one to approximate a composition of flows of the form (3) by the
flow of a single vector field which, moreover, is given by a linear combination of
the $X_w$.

**Notation.** 1. For $M \in \mathbb{N}$ let $W_M \subset W$ denote the collection of all words of
length at most $M$.

2. Given $\delta = (\delta_1, \ldots, \delta_k)$ and $n = (n_1, \ldots, n_k) \in (0, \infty)^k$ we define
$$\delta^n := \prod_{j=1}^{k} \delta_j^{n_j}.$$ 

**Proposition 1** (Baker-Campbell-Hausdorff Formula). There exist a sequence
$\{c_w\}_{w \in W}$ of real numbers such that for all $M \in \mathbb{N}$,
$$e^{t_1 \delta_1 X_1} \circ \cdots \circ e^{t_k \delta_k X_k}(x) = e^{P_M(t\delta)}(x) + O(|\delta|^M+1)$$ 
as $\delta \to 0$, where
$$P_M(t\delta) := \sum_{w \in W_M} c_w(t\delta)^{\deg_w} X_w.$$ 

For example, if $k = 3$ and $M = 2$, then
$$P_2(\delta) = \delta_1 X_1 + \delta_2 X_2 + \delta_3 X_3 + \frac{1}{2} (\delta_1 \delta_2 [X_1, X_2] + \delta_1 \delta_3 [X_1, X_3] + \delta_2 \delta_3 [X_2, X_3]).$$

This leads to the heuristic prediction that if we start at $x$ and, say, flow $\delta_1 \delta_2$
in the $X_{12}$ direction, then we should still remain in some (dilate) of the ball
$B(x; \delta)$. More generally, a dilate of $B(x; \delta)$ should contain all points reached by
flowing $\delta^w$ in the $X_w$ direction for some $w \in W$. Thus $B(x; \delta)$ should resemble
the convex hull of the set of points
$$\{x \pm \delta^{\deg_w} X_w(x) : w \in W\}.$$ 
In particular, for any $n$-tuple of words $I = (w_1, \ldots, w_n)$, (some dilate of) the
parallelepiped whose sides are given by the vectors $\{\delta^{\deg(w_j)} X_{w_j}(x)\}_{j=1}^{n}$ should
lie in $B(x; \delta)$ and hence we arrive at the estimate
$$|B(x; \delta)| \gtrsim \delta^{\deg(I)} |\lambda_I(x)|,$$

¹It is stressed that, without certain assumptions on the $\delta_j$, the heuristic $|B_{1/2}(0, \delta)| \sim |B(0, \delta)|$ may fail [4]. This leads to certain “non-degeneracy” conditions being imposed on the
parameters, as in [6, 7].
where we recall $\lambda_I(x) := \det(X_{w_1}(x) \ldots X_{w_n}(x))$. Suppose there are a number of tuples $I$ for which $\lambda_I(x) \neq 0$. To maximise the right-hand side of (5) one chooses $I$ minimal in the ordering $\preceq$ introduced in the last lecture. Amongst minimal $I$, the choice is then made according to the relative sizes of the $\delta_j$.

Thus, to accurately estimate the volume of $B(x; \delta)$ for any value of $\delta$ one needs to consider $\delta^{\deg(I)|\lambda_I(x)}$ for a range of $I$. In particular, this leads to the heuristic

$$|B(0; \delta)| \sim \sum_I \delta^{\deg(I)|\lambda_I(0)}$$

where the sum is over some finite collection of (minimal) Hörmander-tuples. Some work is required to make these arguments precise, but we will temporarily take (6) for granted.

Testing the estimate (2) with $U := B(0; \delta)$, it follows

$$\sum_I \delta^{\deg(I)} \gtrsim \delta^{b(p)}$$

for all $0 < \delta \lesssim 1$ sufficiently small. For such an inequality to hold, $b(p)$ must lie in a certain polytope which we define presently.

**Definition 2.**

1. For any set $B \subseteq \mathbb{N}_0^k$ define the polytope

$$\mathcal{P}(B) := \text{conv} \left( \bigcup_{b \in B} ([0, \infty)^k + \{b\}) \right).$$

2. For $x_0 \in \Sigma$ define the Newton polytope $\mathcal{P}_{x_0}$ associated to $X_\pi$ at $x_0$ to be

$$\mathcal{P}_{x_0} := \mathcal{P} \left( \{ \deg(I) : \lambda_I(x_0) \neq 0 \} \right).$$

Note in particular,

$$\mathcal{P}_0 := \mathcal{P} \left( \{ \deg(I) : I \text{ is a Hörmander-tuple } \} \right).$$

Thus, our heuristic argument leads to the following conjecture.

**Conjecture 2.** Let $(\Sigma, \{M_j\}_{j=1}^k, \{\pi_j\}_{j=1}^k)$ satisfy the Hörmander condition and $p_j \in [1, \infty]$ with $\sum_{j=1}^k 1/p_j > 1$. Then $T$ is strong-type $(p_1, \ldots, p_k)$ if and only if $b(p) \in \mathcal{P}_0$.

In the following section we will discuss some (very significant) progress on this conjecture, principally due to Tao and Wright [7].

### 3 Main theorem

In [6, 7] the above Conjecture 2 was almost proven.

**Theorem 3** ([6, 7]). For $(\Sigma, \{M_j\}_{j=1}^k, \{\pi_j\}_{j=1}^k)$ and $p_j$ as in Conjecture 2, $T$ is strong-type $(p_1, \ldots, p_k)$ if $b(p)$ lies in the interior of $\mathcal{P}_0$. If $b(p) \notin \mathcal{P}_0$, then $T$ is not restricted $(p_1, \ldots, p_k)$.

Tao and Wright [7] established the case $k = 2$ of the theorem; their methods were modified by Stovall [6] to generalise to $k \geq 2$.

There are numerous examples of operators and forms to which Theorem 3 applies.
Figure 1: A sketch of the Newton polytope $\mathcal{P}_0$ associated to the operator $A_\gamma$ where $\gamma : (-1,1) \to \mathbb{R}^d$ is a non-degenerate curve. The dots correspond to the degrees of minimal Hörmander-tuples.

**Example** (Averages over space curves - non-degenerate case). In the previous lecture it was shown for a non-degenerate curve $\gamma : (-1,1) \to \mathbb{R}^d$,

$$I := (1, 2, 21w_3, \ldots, 21w_d)^t$$

is a minimal Hörmander-tuple for any choice of $w_3, \ldots, w_d \in \{1, 2\}$. The extreme cases occur when all the $w_j$ are chosen to be either 1 or 2 and in these cases the resulting Hörmander-tuples $I_1$ and $I_2$ have degrees

$$\deg(I_1) = \left(1 + \frac{d(d-1)}{2}, d\right) \quad \text{and} \quad \deg(I_2) = \left(d, 1 + \frac{d(d-1)}{2}\right),$$

respectively. It is easy to see that $\mathcal{P}_0$ is therefore the polytope generated by these two points (see Figure 1). By applying Theorem 3 together with duality and some simple algebra on the various exponents one may determine the almost sharp range for which the averaging operator $A_\gamma$ associated to $\gamma$ is bounded $L^p - L^q$. In particular, Theorem 3 implies $A_\gamma$ is strong-type $(p, q)$ whenever $(1/p, 1/q)$ lies in the interior of the trapezium given by the closed, convex hull of the points

$$\{(0,0), (1,1), (1/p_d, 1/q_d), (1/q_d', 1/p_d')\}$$

where $p_d := (d+1)/2$ and $q_d := d(d+1)/2(d-1)$. This, up to endpoints, recovers Christ’s theorem discussed in previous lectures.

**Example** (Averages over space curves - degenerate case). Theorem 3 is robust enough to deal with possibly degenerate curves. We say a curve $\gamma$ is type $N$ if the torsion

$$L_\gamma(t) := \det(\gamma'(t) \ldots \gamma^{(d)}(t))$$

vanishes at most to order $N$ at the origin. Notice $\gamma$ is non-degenerate if and only if it is type 0. If $\gamma$ is not type $N$ for any $N \in \mathbb{N}$, then the Hörmander condition is not satisfied and the associated averaging operator $A_\gamma$ enjoys no non-trivial estimates (see the last lecture). Otherwise, if $\gamma$ is type $N$, then Theorem 3...
implies $A$, is bounded from $L^p - L^q$ for all $(1/p, 1/q)$ belonging to the interior of the trapezium given by the closed, convex hull of the points 

$$\{(0,0), (1,1), (1/p_d,N, 1/q_d,N), (1/q_d,N, 1/p_d,N)\}$$

where 

$$p_{d,N} := \frac{N + \frac{d(d+1)}{2}}{d} \quad \text{and} \quad q_{d,N} := \frac{N + \frac{d(d+1)}{2}}{d-1}.$$ 

As a concrete example, consider the curve 

$$\gamma(t) := (t^{a_1}, \ldots, t^{a_d})$$ 

where $a_1 < \cdots < a_d$ are distinct positive integers. One can show $\gamma$ is type $N := |a| - d(d+1)/2$ where $|a| := \sum_{j=1}^d a_j$. Thus, 

$$p_{d,N} = \frac{|a|}{d} \quad \text{and} \quad q_{d,N} = \frac{|a|}{d-1},$$

and we have $L^p - L^q$ boundedness in the trapezoidal region described above. It is remarked that almost sharp endpoint results for this class of operators were obtained by Gressman [5]. Nevertheless, Theorem 3 appears to give the best known results for more general degenerate curves in high dimensions.

**Example** (A non-degenerate example in $\mathbb{R}^3$). We have seen that in the translation-invariant case, described in the previous examples, the non-degenerate case is given by averaging operators $A$ associated to curves of non-vanishing torsion such as the moment curve $h(t) := (t, \ldots, t^d)$. It is interesting to ask whether, for a given dimension $n = d + 1$, these examples are non-degenerate over the wider class of all generalised Radon transforms. That is, does there exist a (non-zero) Radon transform $R$ acting on $\mathbb{R}^d$ which is bounded on a wider range of $L^p - L^q$ than $A$? If so, what is the widest range of estimates $R$ can enjoy?

By Theorem 3 we can rephrase this problem in terms of finding an operator $R$ whose associated vector fields satisfy the Hörmander condition with minimal $n$-tuples.

Let $A$ denote the averaging operator associated to a non-degenerate curve in each of the respective dimensions.

- When $d = 2$, since no word can repeat, the minimal possible Hörmander-tuple is given by $(1,2,12)$. This is a Hörmander-tuple for $A$ and so $A$ is non-degenerate over the whole class of Radon transforms on $\mathbb{R}^2$.

- Similarly, when $d = 3$ the minimal possible Hörmander-tuples are given by $(1,2,12,121)$ and $(1,2,12,122)$. These are Hörmander-tuples for the averaging operator $A$ and so $A$ is non-degenerate with respect to the whole class of Radon transforms on $\mathbb{R}^3$.

- When $d = 4$ something interesting happens. For the averaging operator the minimal Hörmander-tuples are given by $(1,2,12,12w_1,12w_2w_3)$ for any choice of $w_1, w_2, w_3 \in \{1, 2\}$. However, potentially these tuples are not minimal over all transforms $R$ since it may be possible to find an

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2We could also have $(1,2,21)$ but since $X_{12} = -X_{21}$ this is essentially the same as the previous tuple.
operator for which \((1,2,12,121,122)\) is a Hörmander-tuple. That is, we wish to find an operator \(R\) with associated vector field \(X_1, X_2\) such that
\[
\{X_1, X_2, X_{12}, X_{121}, X_{122}\}
\]
span the tangent space at 0.

Consider the family of curves in \(t \mapsto \gamma(x,t)\) in \(\mathbb{R}^4\) given by
\[
\gamma(x,t) := (x_1 + t, x_2 + t^2, x_3 + t^3, x_4 + t^4 + x_2 t)
\]
and for some smooth cut-off function \(a\) define
\[
Rf(x) := \int f(\gamma(x,t)) a(x,t) dt.
\]

Notice this operator does not belong to the class of translation-invariant examples described above since here the family of curves is not given by translating single curve. Taking the projections \(\pi_j : \mathbb{R}^5 \to \mathbb{R}^4\) to be
\[
\pi_1(x,t) := \gamma(x,t)
\]
and \(\pi_2(x,t) := x\) one easily deduces that the associated vector fields are given by
\[
X_1 := -\partial_{x_1} - 2t \partial_{x_2} - 3t^2 \partial_{x_3} - (4t^3 - 2t^2) \partial_{x_4} + \partial_t, \quad X_2 := \partial_t.
\]
The higher-order commutators are then computed to be:
\[
X_{12} = 2 \partial_{x_2} + 6t \partial_{x_3} + (12t^2 - 4t) \partial_{x_4},
\]
\[
X_{121} := -6 \partial_{x_3} - (24t - 2) \partial_{x_4}, \quad X_{122} := -6 \partial_{x_3} - (24t - 4) \partial_{x_4},
\]
\[
X_{1211} = X_{1212} = X_{1221} = X_{1222} = 24 \partial_{x_4}
\]
and \(X_w = 0\) for any other choice of word \(w\). Note that the minimal collection of vector fields (7) span the tangent space to the origin. Furthermore, the degrees of all Hörmander-tuples in this example are \((7,4), (6,5), (5,5), (5,6), (4,7)\). Applying Theorem 3 (and ignoring endpoints) we see \(R\) is bounded on a larger region than \(A\). In particular, \(R\) is strong-type \((p,q)\) for \((1/p, 1/q)\) lying in interior of the pentagon with vertices
\[
(0,0), \left(\frac{4}{10}, \frac{3}{10}\right), \left(\frac{5}{9}, \frac{4}{9}\right), \left(\frac{7}{10}, \frac{6}{10}\right), (1,1)
\]
whilst \(A\) is strong-type \((p,q)\) for \((1/p, 1/q)\) lying in the trapezium with vertices
\[
(0,0), \left(\frac{4}{10}, \frac{3}{10}\right), \left(\frac{7}{10}, \frac{6}{10}\right), (1,1)
\]

Example (Convolution on the Heisenberg group). Consider the family of curves
\[
\gamma(x,t) := x \cdot (t, t^2, ct^3)
\]
where \(\cdot\) denotes Heisenberg group multiplication and \(c \in \mathbb{R}\) is a fixed constant. Explicitly,
\[
\gamma(x,t) = (x_1 + t, x_2 + t^2, x_3 + ct^3 + \frac{1}{2}(x_1 t^2 - x_2 t)).
\]
If $c \notin \{1/6, -1/6\}$, then the operator $R$ is strong-type $(p,q)$ whenever $(1/p, 1/q)$ lies in the interior of the (set bounded by the) bold trapezium. On the other hand, if $c = 1/6$, then $R$ is not restricted weak-type outside the triangle with vertices $(0,0), (1,1), (1/2,1/3)$, formed by introducing the dashed line. A similar statement is true when $c = -1/6$, but in this case the triangle has vertex $(2/3,1/2)$ rather than $(1/2,1/3)$.

Letting $a$ be a smooth cut-off function we define

$$Rf(x) := \int f(\gamma(x,t))a(x,t)dt$$

and observe the associated vector fields are in this case

$$X_1 := -\partial_{x_1} - 2t\partial_{x_2} - ((3c + 1/2)t^2 - x_2/2 + x_1t)\partial_{x_3} + \partial_t, \quad X_2 := \partial_t.$$ 

It follows

$$X_{12} = 2\partial_{x_2} + ((6c + 1)t + x_1)\partial_{x_3}, \quad X_{122} = -(6c + 1)\partial_{x_3}, \quad X_{121} = (1 - 6c)\partial_{x_3}$$

and all other iterated commutators are zero. Thus the possible contenders for Hörmander tuples are

$$I_1 := (1,2,12,121) \quad \text{and} \quad I_2 := (1,2,12,122).$$

Notice if $c \notin \{1/6, -1/6\}$, then both $I_1$ and $I_2$ are Hörmander-tuples. However, if $c = -1/6$ (respectively, $c = 1/6$), then $I_1$ (respectively $I_2$) is the only Hörmander-tuple. It follows that the range of $L^p - L^q$ estimates for $R$ depends on the value of the constant $c$ (see Figure 2).

**Example** (Non-degenerate Generalised Loomis-Whitney Inequalities). We consider the case of Theorem 3 where $k = n$. In particular, let $(\Sigma, \{M_j\}_{j=1}^n, \{\pi_j\}_{j=1}^n)$ be an $n$-ensemble where we suppose that the associated vector fields satisfy

$$\det(X_1(0) \cdots X_n(0)) \neq 0.$$
Thus the Hörmander condition is satisfied with minimal Hörmander-tuple $I := (1, 2, \ldots, n)$ and we have the estimate
\[
\left| \int_{\mathbb{R}^n} \prod_{j=1}^{n} f_j \circ \pi_j(x) a(x) \, dx \right| \lesssim \prod_{j=1}^{n} \|f_j\|_{L^{n-1+\epsilon}(\mathbb{R}^d)}
\]
for all $\epsilon > 0$ and $a$ of sufficiently small support. Once again, endpoint results (corresponding to $\epsilon = 0$) are known \[1, 2\]. We will discuss this example in more detail in the following section.

4 Non-degenerate Generalised Loomis-Whitney Inequalities

As in the previous lecture series, the general strategy used to establish the estimate (2) (for suitable values of $p$) is to repeatedly refine the set $U$ until we arrive at a subset for which (2) is easily seen to be true. In particular, we shave off parts of the set until it “looks like” a Carnot-Carathéodory ball, which we already know how to estimate (at least heuristically). To illustrate the process we consider the special situation of the Loomis-Whitney inequalities introduced above: in this case the arguments are substantially simplified. Recall, in this example we consider an $n$-ensemble $(\Sigma, \{M_j\}_{j=1}^{n}, \{\pi_j\}_{j=1}^{n})$ where we suppose that the associated vector fields satisfy
\[
\det(\{X_i(0)\ldots X_n(0)\}) \neq 0.
\]
Thus the Hörmander condition is satisfied with Hörmander-tuple $I := (1, 2, \ldots, n)$ and the above condition may be rewritten as $\lambda_I(0) \neq 0$. Here we will actually prove a strengthened version of the result given by Theorem 3.

**Theorem 4** (Bennett, Carbery, Wright \[2\]). $(\Sigma, \{M_j\}_{j=1}^{n}, \{\pi_j\}_{j=1}^{n})$ is restricted $(n-1, \ldots, n-1)$.

**Remark 1.** In \[2\] a strong-type estimate was established as a consequence of Theorem 4 by applying the tensor power trick.\(^3\) Furthermore, the dependence of the constant $|\lambda_I(0)|$ was accurately computed; in particular, one has
\[
\left| \int_{\mathbb{R}^n} \prod_{j=1}^{n} f_j \circ \pi_j(x) a(x) \, dx \right| \leq C_n |\lambda_I(0)|^{-1/(n-1)} \prod_{j=1}^{n} \|f_j\|_{L^{n-1}(\mathbb{R}^d)}.
\]

Compare this inequality (and especially the constant which appears) with the linear Loomis-Whitney inequalities discussed in the previous lecture. We will only prove the weaker statement Theorem 4 as this suffices to demonstrate the rudiments of the refinement method.

**Proof (of Theorem 4).** Fix $U \subset \Sigma$ with positive measure and for a fixed point $x \in U$ consider the mapping
\[
\Phi_n(t_1, \ldots, t_n) := e^{t_1 X_1} \circ \cdots \circ e^{t_n X_n} x.
\]

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\(^3\)For a description of this method and numerous applications see, for instance, \[3\] or http://www.tricki.org/article/The_tensor_power_trick.
Provided \( U \) belongs to a sufficiently small neighbourhood of the origin\(^4 \) the following hold:

1) \( \Phi_n \) is injective when \(|t| \lesssim 1\);

2) If \( J_{\Phi_n} \) denotes the Jacobian determinant of the mapping \( \Phi_n \), then
\[
|J_{\Phi_n}(t)| \gtrsim |\lambda_I(0)|
\]
for all \(|t| \lesssim 1\).

3) \( \pi_j^{-1}(\{\pi_j(x)\}) \cap U = \{e^{iX_j}x : |t| \lesssim 1\} \cap U \) for \( 1 \leq j \leq n \).

These properties follow from the fact \( J_{\Phi_n}(0) = \lambda_I(x) \), the hypothesis \( \lambda_I(0) \neq 0 \) together with some calculus. Although the details are not entirely straightforward, we omit them here referring to [2] for a more thorough exposition.

We assume \( U \) is lies in a sufficiently small neighbourhood of \( 0 \) so that 1) - 3) are true for any \( x \in U \). Rephrasing the estimate as a lower bound on \(|U|\) as in (2), it suffices to show
\[
|U| \gtrsim |\lambda_I(0)| \prod_{j=1}^n \alpha_j. \tag{8}
\]

To prove (8) we begin by constructing a sequence of refinements \( \{U_j\}_{j=0}^n \) of \( U \). This is done recursively so that each \( U_j \) satisfies:

i) \( |U_j| \geq 2^{-j}|U| \);

ii) If \( x_0 \in U_j \), then \( \{|t| \in \mathbb{R} : |t| \lesssim 1 \text{ and } e^{iX_j}x_0 \in U_{j-1}\} \gtrsim \alpha_j \).

Indeed, let \( U_0 := U \) (property ii) is vacuous when \( j = 0 \) and suppose for some \( 1 \leq j \leq n \), \( \{U_i\}_{i=0}^{j-1} \) have already been constructed. Define \( E_j := \pi_j(U_{j-1}) \) by
\[
E_j := \left\{ y \in \pi_j(U) : \int_{\pi_j^{-1}(\{y\}) \cap U_{j-1}} \frac{dH^1(x)}{X_j(x)} \geq 2^{-j}\alpha_j \right\}
\]
and take \( U_j := U_{j-1} \cap \pi_j^{-1}(E_j) \). Thus \( U_j \) is the collection of all points of \( U_{j-1} \) which belong to “large” \( \pi_j \)-fibres. As such, \( U_j \) is easily seen to have measure comparable to \( U_{j-1} \). In particular, since the definition of \( U_j \) is fibre-wise, it follows whenever \( y \in E_j \),
\[
\int_{\pi_j^{-1}(\{y\}) \cap U_{j}} \frac{dH^1(x)}{|X_j(x)|} = \int_{\pi_j^{-1}(\{y\}) \cap U_{j-1}} \frac{dH^1(x)}{|X_j(x)|}
\]
and so,
\[
|U_{j-1}| = \int_{\pi_j(U_{j-1}) \cap E_j} \left( \int_{\pi_j^{-1}(\{y\}) \cap U_{j-1}} \frac{dH^1(x)}{|X_j(x)|} \right) dy
\]
\[
\leq \int_{E_j} \int_{\pi_j^{-1}(\{y\}) \cap U_j} \frac{dH^1(x)}{|X_j(x)|} dy + 2^{-j}\alpha_j|\pi_j(U)|
\]
\[
\leq |U_j| + 2^{-j}|U_{j-1}|.
\]

\(^4\)This neighbourhood depends on the values \(|\lambda_I(0)|\), \( \|\pi_j\|_{C^3} \) and \( n \). In [1] an alternative proof of the result is given which provides a more natural dependence on the regularity of the \( \pi_j \).
From this one deduces \(|U_j| \geq 2^{-1}|U_{j-1}| \geq 2^{-j}|U|\) and so property i) holds. We turn to showing property ii) is satisfied by \(U_j\). Indeed, for \(x_0 \in U_j\) let \(y_0 := \pi_j(x_0)\) and recall \(e^{tX_j}x_0 \in \pi^{-1}\{\{y\}\}\) whenever \(|t| \lesssim 1\). Thus,
\[
\{|t \in \mathbb{R} : |t| \lesssim 1 \text{ and } e^{tX_j}x_0 \in U_{j-1}\|}
\]
can be expressed as
\[
\int_{|t| \lesssim 1} \chi_{U_{j-1}}(e^{tX_j}x_0) \, dt = \int_{|t| \lesssim 1} \chi_{\pi_j^{-1}\{\{y_0\}\} \cap U_{j-1}}(e^{tX_j}x_0) \, dt
\]
by the change of variables formula and the definition of \(U_j\).
This concludes the construction of the refinements \(\{U_j\}_{j=0}^n\). Choose \(x_0 \in U_n\), noting the set is non-empty since \(|U_n| \geq 2^{-n}|U| > 0\). Define a sequence of functions
\[
\Phi_j(t_{n-j+1}, \ldots, t_n) := e^{t_{n-j+1}X_{n-j+1}} \cdots e^{t_nX_n}x_0.
\]
We construct a sequence of sets (a parameter tower in the language of the previous lectures) \(\{\Omega_j\}_{j=1}^n\), each with the following properties:

a) \(\Omega_j \subset \mathbb{R}^j\);

b) \(\Phi_j(\Omega_j) \subset U_{n-j}\).

To do this we again use a recursive definition. Let
\[
\Omega_1 := \{t_n \in \mathbb{R} : |t_n| \lesssim 1 \text{ and } e^{t_nX_n}x \in U_{n-1}\}
\]
and suppose for some \(1 \leq j \leq n - 1\) the sets \(\{\Omega_j\}_{j=1}^n\) have been constructed. For \(t = (t_{n-j+1}, \ldots, t_n) \in \Omega_j\) define
\[
\Omega_{j+1}(t) := \{t_{n-j} \in \mathbb{R} : |t_{n-j}| \lesssim 1 \text{ and } \Phi_{j+1}(t_{n-j}, t) \in U_{n-j-1}\}
\]
\[
= \{t_{n-j} \in \mathbb{R} : |t_{n-j}| \lesssim 1 \text{ and } e^{t_{n-j}X_{n-j}}\Phi_j(t) \in U_{n-j-1}\}.
\]
Since \(\Phi_{j-1}(t) \in U_{n-j-1}\) by property b) of the construction, it follows \(|\Omega_{j+1}(t)| \gtrsim \alpha_{n-j}\). Define
\[
\Omega_{j+1} := \{(t_{n-j}, \ldots, t_n) : t := (t_{n-j+1}, \ldots, t_n) \in \Omega_j \text{ and } t_{n-j} \in \Omega_{j+1}(t)\}
\]
This concludes the construction of the parameter tower. Recall, since we have restricted our analysis to a small neighbourhood of the origin, the mapping \(\Phi_n\) is guaranteed to be injective. Applying the change of variables formula it follows
\[
|U| \geq |\Phi_n(\Omega_n)| \geq \int_{\Omega_n} |J_{\Phi_n}(t)| \, dt
\]
where \(J_{\Phi_n}\) is the Jacobian determinant of the mapping \(\Phi_n\). In addition, by 2) above
\[
|J_{\Phi_n}(t)| \gtrsim |\lambda_t(0)| \quad \text{for } t \in \Omega_n
\]
and so \(|U| \gtrsim |\lambda_t(0)||\Omega_n|\). Applying Fubini’s theorem one may readily deduce \(|\Omega_n| \gtrsim \prod_{j=0}^{n-1} \alpha_j\) and so (8) is established and this concludes the proof.
References


