LECTURE 4: EQUIVALENCE OF MULTI-LINEAR AND LINEAR
$$\ell^2$$-DECOUPLING INEQUALITIES

JONATHAN HICKMAN AND MARCO VITTURI

1. Multi-linear decoupling inequalities

Armed with the tools developed in the previous lecture, we now embark on the task of proving the $$\ell^2$$-decoupling theorem, the statement of which we presently recall.

**Theorem 1** ($$\ell^2$$-decoupling theorem). For $$2 \leq p \leq \infty$$ the decoupling inequality

$$\|f\|_{L^p(B_R)} \lesssim R^\alpha(p)\|f\|_{L^p, n^{-1}(\mathbb{R}^n)}$$

holds for all Schwartz functions $$f$$ with Fourier support in $$\mathcal{N}_{\mathbb{R}^{-1}}(P^{n-1})$$.

Note here we’re working with a spatially-localised version of the decoupling inequality (which can be shown to be equivalent to the global version via uncertainty principle techniques) and $$\alpha(p)$$ is the ‘decoupling exponent’ introduced in earlier lectures and defined by

$$\alpha(p) := \begin{cases} 0 & \text{if } 2 \leq p \leq 2(n+1)/(n-1) \\ (n-1)/4 - (n+1)/2p & \text{if } 2(n+1)/(n-1) < p \end{cases}.$$ 

We begin by noting that this inequality (like so many of the estimates we’ve encountered) is trivial at small scales.

**Lemma 2** (Trivial decoupling). For $$2 \leq p \leq \infty$$ the decoupling inequality

$$\|f\|_{L^p(B_R)} \lesssim R^{n(1/2 - 1/p)}\|f\|_{L^p, n^{-1}(w_{B_R})}$$

holds for all Schwartz functions $$f$$ with Fourier support in $$\mathcal{N}_{\mathbb{R}^{-1}}(P^{n-1})$$.

**Proof.** By the local Bernstein and local orthogonality inequalities proved in the last lecture, for $$p \geq 2$$ and $$f$$ satisfying the Fourier support condition it follows that

$$\|f\|_{L^p(B_R)} \lesssim \|f\|_{L^2(w_{B_R})} \lesssim \|f\|_{L^2, n^{-1}(w_{B_R})}.$$

By Hölder’s inequality, for $$\theta$$ any $$R^{-1/2}$$-slab

$$\|f_\theta\|_{L^2(w_{B_R})} \lesssim R^{n(1/2 - 1/p)}\|f_\theta\|_{L^p(w_{B_R})},$$

and taking the square sum over the slabs concludes the proof. \(\square\)

In the previous notes we saw how one can apply an ‘induction on scale’ to reduce the proofs of certain restriction estimates to (trivial) small scale cases. The same approach can be applied in the decoupling context. Furthermore, recall as part of this induction procedure (the Bourgain-Guth method) we considered multi-linear estimates which involved restricting Fourier transforms of functions to families of transverse hypersurfaces. In analogy with the development of multi-linear restriction theory, one may formulate a theory of a multi-linear $$\ell^2$$-decoupling inequalities. The multi-linear estimates can then be applied in an induction on scale to prove linear decoupling inequalities: this is exactly how Bourgain and Demeter proceeded in [BD].

1
Theorem 3 (Multi-linear $\ell^2$-decoupling [BD]). Let $\tau_1, \ldots, \tau_n$ be $\omega$-transversal regions of $N_{R^{-1}}(P^{n-1})$. For $p \geq 2$, the inequality
\[
\left\| \prod_{j=1}^n |f_j|^{1/n} \right\|_{L^p(B_R)} \leq \omega^{-O(1)} R^{\alpha(p)} \prod_{j=1}^n \left\| f_j \right\|_{L^p, R^{-1}(\mathbb{R}^n)}^{1/n}
\]
holds for $f_j$ with Fourier support in $\tau_j$, $1 \leq j \leq n$.

We recall that for $f_j$ with Fourier support in $\tau_j$ we have
\[
\left\| f_j \right\|_{L^p, R^{-1}(\mathbb{R}^n)} = \left( \sum_{\theta \cap \tau_j \not\subseteq \emptyset} \left\| f_j, \theta \right\|_{L^p(\mathbb{R}^n)}^2 \right)^{1/2}.
\]

The estimate (1) is certainly redolent of the (reformulation of the) multi-linear restriction estimate: the latter is a very similar expression but with $L^2(\mathbb{R}^n)$-norms appearing on the right-hand side rather than decoupling norms. One can, in addition, easily show that Theorem 3 is equivalent to the following statement involving extension operators.

Theorem 4 (Multi-linear $\ell^2$-decoupling [BD]). Let $\kappa_1, \ldots, \kappa_n$ be $\omega$-transversal caps on $P^{n-1}$. For $p \geq 2$, the inequality
\[
\left\| \prod_{j=1}^n \left| (g_{\kappa_j}(\omega \sigma))^{-1/n} \right| \right\|_{L^p(B_R)} \leq \omega^{-O(1)} R^{\alpha(p)} \prod_{j=1}^n \left( \sum_{\theta \cap \kappa_j \not\subseteq \emptyset} \left\| (g_{\kappa_j}(\omega \sigma))^{-1/2} \right\|_{L^p(w_{R/H})}^2 \right)^{1/2n}
\]
holds whenever each $g_j$ is a smooth function on $\kappa_j$.

The proof of these multi-linear estimates will be postponed until the following lecture; for now we wish to investigate how Theorem 3 can be used to establish linear decoupling estimates. As we’ve seen in the restriction case, the main obstacle one comes up against here is the transversality hypothesis. However, in the context of decoupling theory the Bourgain-Guth method is very effective in dealing with transversality and one can roughly show the following implication.

Proposition 5 ([BD]). Informally, the multi-linear $\ell^2$-decoupling theorem implies the linear $\ell^2$-decoupling theorem; that is:

Theorem 3 $\Rightarrow$ Theorem 1.

Given the discussion of the multi-linear theory in the previous set of notes, the reader should rightly suspect that Proposition 5 significantly reduces the problem of proving Theorem 1.

2. Reduction to multi-linear estimates

In this section we address the relationship between the linear and multi-linear $\ell^2$-decoupling inequalities and, in particular, give a precise statement of Proposition 5. In order to compare the constants involved, we introduce the following notation.

- Let $\mathfrak{D}_{p,n}(R)$ denote the best constant for the linear $\ell^2$-decoupling inequality
\[
\left\| f \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| f \right\|_{L^p, R^{-1}(\mathbb{R}^n)};
\]
that is, $\mathfrak{D}_{p,n}(R)$ is the infimum over all $C$ for which (2) holds for every Schwartz $f$ with $\hat{f}$ supported in $N_{R^{-1}}(P^{n-1})$.

- Similarly, we let $\mathfrak{M}_{p,n}(R, \omega)$ denote the best constant for the multi-linear $\ell^2$-decoupling inequality
\[
\left\| \prod_{j=1}^n |f_j|^{1/n} \right\|_{L^p(B_R)} \leq C \prod_{j=1}^n \left\| f_j \right\|_{L^p, \omega^{-1}(w_{R/H})}^{1/n};
\]
that is, $\mathcal{M}_{p,n}(R,\omega)$ is the infimum over all $C$ for which (3) holds for all $\omega$-transverse regions $\tau_1, \ldots, \tau_n$ of $N_{R^{-1}}(P^{n-1})$ and all Schwartz functions $f_j$ with $f_j$ supported in $\tau_j$.

**Remark 6.** Using the localisation techniques described in earlier notes one can show the linear and multi-linear decoupling estimates admit various reformulations whose best constants (for $p$ and $n$ fixed) are all comparable. In particular, if we consider the estimates (stated in terms of their respective best constants)

\[\|f\|_{L^p(\mathbb{R}^n)} \leq \mathcal{D}_{p,n}(R)\|f_0\|_{L^{p,n-1}(\mathbb{R}^n)}, \quad \|f\|_{L^p(B_R)} \leq \mathcal{D}_{p,n}^1(R)\|f_0\|_{L^{p,n-1}(w_{B_R})},\]

and

\[\|(g\theta d\sigma)^\ast\|_{L^p(B_R)} \leq \mathcal{D}_{p,n}^3(R) \left( \sum_{\theta \in R^{-1/2} \cap \mathcal{P}} \|g\theta d\sigma\|_{L^p(w_{B_R})}^2 \right)^{1/2},\]

then it follows that

\[\mathcal{D}_{p,n}(R) \sim \mathcal{D}_{p,n}^1(R) \sim \mathcal{D}_{p,n}^2(R) \sim \mathcal{D}_{p,n}^3(R).\]

Similar statements can be made in the multi-linear case (as we’ve already seen from the discussion of the equivalence of Theorems 3 and 4).

The rigorous version of the statement anticipated in Proposition 5 is given by the following.

**Proposition 7.** Suppose that either $n - 2$ or $n \geq 3$ and $\mathcal{D}_{p,n-1}(R) \leq 1$. For $2 \leq p$ and any $\varepsilon > 0$ one can choose $0 < \omega \leq 1$ sufficiently small depending on $\varepsilon$ such that

\[\mathcal{D}_{p,n}(R) \leq \omega^{-O(1)} R^\varepsilon \mathcal{M}_{p,n}(R,\omega),\]

for all $R \geq 1$.

This proposition will facilitate an inductive procedure on the dimension $n$ and thereby reduce the problem of bounding $\mathcal{D}_{p,n}$ to controlling the $n - 2$ case and bounding $\mathcal{M}_{p,n}$.

### 3. Applying the Bourgain-Guth method

Proposition 7 can be established via the Bourgain-Guth method and much of our presentation now mirrors that of the previous lecture notes. We’ll see, however, that there are additional features present in the decoupling context which will allow for a ‘cleaner’ analysis than what we saw in the case of restriction.

#### 3.1. Setting up the induction.

The argument is based upon another induction on scale and here we’ll compare behaviour at a large scale $R$ with the same inequality at some smaller scale $R/K$ for some suitable $1 \ll K \ll R$.

Fix some $f \in \mathcal{F}(\mathbb{R}^n)$ with Fourier support in $N_{R^{-1}}(P^{n-1})$. In order to facilitate the passage between different scales we consider the frequency restriction of $f$ to small subsets of its frequency support. In particular, cover $N_{R^{-1}}(P^{n-1})$ with a family of finitely-overlapping $K^{-1}$-regions $\{\tau\}$, noting such a family must consist of $O(K^2)$ members. Let $f_\tau$ denote a frequency restriction of $f$ to some subset of $\tau$ so that

\[f = \sum_{\tau:K^{-1} \text{-reg.}} f_\tau.\]

We will need to be able to relate estimates at different scales; this is achieved using a variant of the parabolic rescaling lemma we saw in earlier notes.
Lemma 8 (Parabolic rescaling). If $g$ is a (say) Schwartz function supported in a $\rho^{1/2}$-region of $N_{R^{-1}}(P^{n-1})$, with $\rho < R^{-1}$, then
\[ \|g\|_{L^p(B_R)} \lesssim \mathcal{D}_{p,n}(\rho R) \|g\|_{L^{p,n}(-\omega R)}^{1/2} \|
\]

As before, the proof is nothing but a change of variables. The simple details are postponed until a later subsection.

3.2. The $n - 2$ case. We temporarily consider the situation when $n = 2$, which is rather straightforward. As in the previous lecture, we borrow the ‘broad’ terminology from [Gut].

Definition. For $0 < \lambda < 1$ and $x \in B_R$ we say $x$ is $\lambda$-broad for $f$ if
\[ \max_{\tau : K^{-1}-\text{reg.}} |f_\tau(x)| \leq \lambda |f(x)|. \]

We define the $\lambda$-broad part of $f$ to be the function $B_\lambda f$ which agrees with $f$ on the set of broad points and vanishes everywhere else.

We remark that although the notion of broad points played an important rôle in the previous lecture, here it will only really be useful in the $n - 2$ case.

By the triangle inequality it follows that
\[ \|f\|_{L^p(B_R)} \leq \|B_\lambda f\|_{L^p(B_R)} + \lambda^{-1} \left( \sum_{\tau : K^{-1}-\text{reg.}} \|f_\tau\|_{L^p(B_R)} \right)^{1/2} \quad (4) \]

and one may apply the ‘bilinear reduction’ to obtain the pointwise estimate
\[ |B_\lambda f(x)| \leq K^2 \sum_{\tau_1, \tau_2 : K^{-1}-\text{reg.}, \text{dist}(\tau_1, \tau_2) \geq K^{-1}} \prod_{j=1}^{2} \|f_{\tau_j}(x)\|^{1/2}, \quad (5) \]

provided $0 < \lambda < 1$ is chosen to be suitably small.

At this point we can easily deduce the $n - 2$ case of Proposition 7. Indeed, one may apply the bilinear reduction (5) together with parabolic rescaling to deduce
\[ \|f\|_{L^p(B_R)} \leq K^2 \sum_{\tau_1, \tau_2 : K^{-1}-\text{reg.}, \text{dist}(\tau_1, \tau_2) \geq K^{-1}} \prod_{j=1}^{2} \|f_{\tau_j}\|^{1/2}_{L^p(B_R)} + \mathcal{D}_{p,2}(R/K) \|f\|_{L^{p,n}(-\omega R)}^{1/2}. \]

By definition, it follows that whenever $\text{dist}(\tau_1, \tau_2) \geq K^{-1}$ one has
\[ \|f\|_{L^p(B_R)} \leq \sum_{j=1}^{2} \|f_{\tau_j}\|^{1/2}_{L^p(B_R)} \leq M_{p,2}(R, K^{-1}) \prod_{j=1}^{2} \|f_{\tau_j}\|^{1/2}_{L^{p,n}(-\omega R)} \]

and, consequently,
\[ \|f\|_{L^p(B_R)} \lesssim \left( K^4 M_{p,2}(R, K^{-1}) + \mathcal{D}_{p,2}(R/K) \right) \|f\|_{L^{p,n}(-\omega R)}. \]

We therefore have the recurrence relation
\[ \mathcal{D}_{p,2}(R) \lesssim C \left( K^4 M_{p,2}(R, K^{-1}) + \mathcal{D}_{p,2}(R/K) \right) \]

which, after $m$-fold application yields
\[ \mathcal{D}_{p,2}(R) \lesssim K^4 \sum_{j=1}^{m} C^j M_{p,2}(R/K^{j-1}, K^{-1}) + C \mathcal{D}_{p,2}(R/K^m) \]

\[ \lesssim C^m K^4 M_{p,2}(R, K^{-1}) + \mathcal{D}_{p,2}(R/K^m). \]

for any integer $m$. If we choose $m \sim \log_K R$, then the second term can be bounded by an absolute constant via the trivial decoupling lemma and taking $K := \omega^{-1}$ we deduce
\[ \mathcal{D}_{p,2}(R) \lesssim \omega^{-4} R^{\omega(\omega)} M_{p,2}(R, \omega) \]
Proof (of Lemma 10). Observe in a rather straight-forward manner. This extension follows from the classical Bernstein inequality (for $1 \leq p \leq q \leq \infty$).

Remark 9. In dimensions $n \geq 3$ we will not rely on any bilinear estimates and, as mentioned earlier, the isolation of the broad points is somewhat redundant: this is one of the simpler aspects of the Bourgain-Guth method in the decoupling context.\footnote{Recall, for restriction we still used bilinear square function estimates when working over $\mathbb{R}^3$ and so the broad points were important also in higher dimensions.}

3.3. The $n \geq 3$ case. We now turn to the higher dimensional scenario, where it remains to effectively bound the broad contribution. Reduction to the broad contribution offers no particular simplification here and we might as well estimate $\|f\|_{L^p(B_R)}$ directly. For simplicity, we will focus on the $n - 3$ case; the subsequent analysis can be generalised to higher dimensions without too much difficulty.

Note that as each $f_x$ is supported in some $K^{-1}$-ball it is essentially constant on balls $B_K$. As in the previous set of notes, we will carry out a simple geometric analysis to identify the balls $B_K$ for which large contributions arise from either transverse or coplanar planar regions.

We begin by giving a rigorous formulation of the locally-constant heuristic; for technical reasons this will be realised in a slightly different manner to the previous notes. Take a smooth function $\eta \in \mathcal{S}(\mathbb{R}^3)$ such that $\hat{\eta}$ has support in $B(0, 2)$ and $\hat{\eta}(\xi) = 1$ for $\xi \in B(0, 1)$, and set $\eta_K(x) := K^{-3}\hat{\eta}(K^{-1}x)$. Notice that $\eta_K - 1$ on $B(0, K^{-1})$. Then $f_x(\xi) = \hat{f}_x(\xi)\eta_K(\xi - \tau)$, where $\tau$ is the centre of the slab, and therefore

$$|f_x(x)| \leq \int_{\mathbb{R}^3} |f_x(x - z)||\eta_K(z)| \, dz, \quad (6)$$

and this is leads to the familiar interpretation of the heuristic. Here, however, it is useful to work not with (6) but a “strengthened” version of this identity.

Lemma 10. The inequality\footnote{Roughly speaking, it is desirable to have the $1/3$-power appearing owing to the $1/3$-power arising from the tri-linear decoupling theorem.}

$$|f_x(x)| \lesssim \left( \int |f_x(x - z)|^{1/3}|\eta_K(z)|^{1/3} \, dz \right)^3$$

holds for some non-negative power of $K$.

Thus, if $\zeta \geq |\eta|^{1/3}$ is a rapidly decreasing weight concentrated in the unit ball, then we may define

$$c_\tau(x) := (|f_x|^{1/3} \ast \zeta_K(x))^3$$

so that $|f_x(x)| \lesssim c_\tau(x)$. Here we will choose $\zeta$ to be approximately constant on balls of unit radius so that $\zeta_K$ is approximately constant on $K$-balls. The latter can be thought of as a strengthened version of (6) since, by Hölder’s inequality,

$$c_\tau(x) \lesssim |f_x| \ast \zeta_K(x).$$

The proof of Lemma 6 relies on a general form of Bernstein’s inequality, valid for exponents less than 1. In particular, if $0 < p \leq q \leq \infty$ and $g$ is an integrable function on $\mathbb{R}^n$ satisfying $\text{supp} \, \hat{g} \subseteq B_r$, then

$$\|g\|_{L^q(\mathbb{R}^n)} \lesssim r^{n(1/p - 1/q)}\|\hat{g}\|_{L^p(\mathbb{R}^n)}.$$

This extension follows from the classical Berstein inequality (for $1 \leq p \leq q \leq \infty$) in a rather straightforward manner.

Proof (of Lemma 10). Observe

$$|f_x(x)| \leq \|g_x\|_{L^1(\mathbb{R}^3)}$$
where \( g_\tau(z) := f_\tau(x - z)\eta_K(z) \). This function may easily be seen to have Fourier support contained in \( B(0,10) \) and so the proof is concluded by appealing to Bernstein’s inequality with exponents 1 and 1/3.

We now proceed by implementing the geometric analysis we saw in the previous lecture. Given \( x \in B_R \), define

\[
c_\tau(x) := \max_{\tau : K^{-1}-\text{reg.}} c_\tau(x);
\]

and let

\[
T^\tau := \{ \tau : c_\tau(x) > K^{-2}c_\tau(x) \};
\]

this is the set of regions which provide a large contribution to \( \sum c_\tau(x) \). With this set-up we have a dichotomy: for any point \( x \in B_R \) one of the following holds:

1) **Transverse case:** There exist \( \tau \in K^{-2}\)-transverse regions \( \tau_1, \tau_2, \tau_3 \in T^\tau \).

2) **Coplanar case:** There exists a line \( \ell(x) \subseteq \mathbb{R}^2 \) such that if \( \text{dist}(\pi_{\tau}, \ell(x)) > 10^3K^{-1} \), then \( c_\tau(x) \leq K^{-2}c_\tau(x) \).

Let \( E_{\text{trans}} \) denotes the set of all \( x \in B_R \) for which the transverse case holds so that

\[
\|\text{Br}_x f\|_{L^p(B_R)} \leq \|\text{Br}_x f\|_{L^p(E_{\text{trans}})} + \|\text{Br}_x f\|_{L^p(B_R \setminus E_{\text{trans}})}.
\]

We will estimate the two terms separately.

1) **Transverse case.** Here we apply the tri-linear inequality and our argument is very similar to what we encountered in the previous lecture. For \( x \in E_{\text{trans}} \) it follows that

\[
c_\tau(x) \leq c_\tau(x) < K^2 \prod_{j=1}^3 c_{\tau_j}(x)^{1/3}
\]

for every \( K^{-1}\)-region \( \tau \) and, consequently,

\[
|f(x)| \leq \sum_{\tau : K^{-1}-\text{reg.}} |f_{\tau}(x)| \leq K^4 \prod_{j=1}^3 c_{\tau_j}(x)^{1/3} \leq K^4 \sum_{(\tau_j)_j=1}^3 K^{-2}\text{-trans.} \prod_{j=1}^3 c_{\tau_j}(x)^{1/3}.
\]

Raising this to the \( p \)-th power and applying Hölder’s inequality, we have

\[
|f(x)|^p \leq K^{4p} \sum_{(\tau_j)_j=1}^3 \prod_{j=1}^3 |f_{\tau_j}(x)|^{1/3} \cdot \zeta_K(x)^p 
\]

\[
\leq K^{4p} \sum_{(\tau_j)_j=1}^3 \prod_{j=1}^3 \int_{(\mathbb{R}^3)^3} \left( \prod_{j=1}^3 |f_{\tau_j}(x - z_j)| \right)^{p/3} Z_K(z) \, dz,
\]

where \( Z_K(z) = \prod_{j=1}^3 \zeta_K(z_j) \) for \( z = (z_1, z_2, z_3) \in (\mathbb{R}^3)^3 \). Note that this application of Hölder’s inequality is facilitated by our definition of the \( c_\tau \) involving 1/3-powers. Integrating over \( E_{\text{trans}} \), it follows that

\[
\|f\|_{L^p(E_{\text{trans}})} \leq K^{4p} \sum_{(\tau_j)_j=1}^3 \prod_{j=1}^3 \int_{(\mathbb{R}^3)^3} \left( \prod_{j=1}^3 |f_{\tau_j}(x - z_j)| \right)^{1/3} \|L^p(B_R) Z_K(z)\|_{L^p(B_R)} \, dz
\]

Fixing a \( K^{-2}\)-transverse triple \( (\tau_j)_j=1 \), by the multi-linear \( \ell^2\)-decoupling estimate one may bound the corresponding summand from the above expression by

\[
\mathcal{M}_{p,3}(R^{-1}, K^{-2})^p \sum_{j=1}^3 \|f_{\tau_j}(\cdot - z_j)\|_{L^{p,3}(w_{BR})}^{p/3} Z_K(z) \, dz.
\]
Applying Hölder’s and Minkowski’s inequalities in succession then dominates the above by

$$
\mathfrak{M}_{p,3}(R^{-1}, K^{-2})^3 \prod_{j=1}^{3} \sum_{\theta \in \mathcal{R}_{\text{lab}}} \left( \int_{\mathbb{R}^3} \|f(x - z_j)\|_{L^p(w_B)}^p \zeta_K(z_j)dz_j \right)^{2/p} \left( \int_{\mathbb{R}^3} \|f(x - z)\|_{L^p(w_B)}^p \zeta_K(z)dz \right)^{1/p}.
$$

It is now an easy observation that

$$
\int_{\mathbb{R}^3} \|f(x - z)\|_{L^p(w_B)}^p \zeta_K(z)dz = \|f\|_{L^p(w_B)}^p,
$$

since $w_B \ast \zeta_K$ is a rapidly decreasing weight concentrated in the ball $B_R$ for $K \ll R$. Putting everything together, we have proven

$$
\|f\|_{L^p(E_{\text{trans}})} \lesssim K^{3\mathfrak{M}_{p,3}(R^{-1}, K^{-2})} \left( \sum_{\tau \in \mathcal{R}} \|f\|_{L^p(w_B)}^p \right)^{1/p},
$$

from which one concludes that

$$
\|f\|_{L^p(E_{\text{trans}})} \lesssim K^{10\mathfrak{M}_{p,3}(R^{-1}, K^{-2})}\|f\|_{L^p(w_B)}
$$

since there are $O(K^6)$ terms in the sum over the transversal triplets.

2) Coplanar case. We now consider the contribution arising from non-transverse points, which satisfy the coplanar condition 2). Now the dominant contributions cluster around a parabola embedded in $P^2$ and, as in the case of the proof of the restriction estimate from the last lecture, we can reduce to a lower dimensional estimate. For decoupling inequalities, however, we will see that this reduction can be made somewhat tighter than in the restriction case.

Cover $B_R \setminus E_{\text{trans}}$ by finitely overlapping balls $B_K$ which, without loss of generality, have non-trivial intersection with $B_R \setminus E_{\text{trans}}$. Using the locally-constant property and the definition of the co-planar case, we deduce for each such ball $B_K$ there exists a line $\ell(B_K)$ such that if $\text{dist}(\tau, \ell(B_K)) > 10^4K^{-1}$, then $c_\tau(x) \lesssim K^{-2}c_\tau(x)$ uniformly for all $x \in B_K \setminus E_{\text{trans}}$. Denoting by $\mathcal{L}(B_K)$ the fattened line $N_{10^4K^{-1}}(\ell(B_K))$, we estimate

$$
|f(x)| \lesssim \left( \sum_{\tau \in \mathcal{N}_{10^4K^{-1}}} \left( \sum_{\tau \in \mathcal{L}(B_K)} c_\tau(x) \right)^2 \right)^{1/2}.
$$

We will consider the two terms of the right-hand side separately. The latter summand is easily dealt with. Taking the $L^p$-norm over $B_K$ we note that

$$
\left( \sum_{\tau \in \mathcal{N}_{10^4K^{-1}}} \left( \sum_{\tau \in \mathcal{L}(B_K)} c_\tau(x) \right)^2 \right)^{1/2} \lesssim \left( \sum_{\tau \in \mathcal{N}_{10^4K^{-1}}} \|c_\tau\|_{L^p(w_B)}^2 \right)^{1/2}
$$

The latter estimate is due to the simple inequality

$$
\|c_\tau\|_{L^p(B_K)} \lesssim \|f_\tau\|_{L^p(w_B)}. 
$$

The right-hand side of (8) can now be easily controlled by summing up over all $B_K$ in the cover and applying a parabolic rescaling argument; we will postpone the details (which are nevertheless very simple) until later in the argument.

We duly turn to investigating the contribution arising from the first summand of the right-hand side of (7), which we note now depends on $B_K$. In particular, we
wish to bound the $L^p$-norm of this function over the associated ball, given by

$$\| f \|_{L^p(B_K)} = \sum_{\tau \in \mathcal{L}(B_K)} \tau \cdot K^{-1 - \reg} \| f \|_{L^p(B_K)}.$$  

(9)

By rotation invariance we assume, for simplicity, that the underlying line $\ell(B_K) = \{(0, \xi_2) : \xi_2 \in \mathbb{R}\}$. We cover the union of those $\tau$’s such that $\tau \in \mathcal{L}(B_K)$ by finitely overlapping ‘regions’ $U$ of length $K^{-1/2}$ and width $10^4K^{-1}$. Note the $U$ are not regions in formal sense to which we are now accustomed.

Recall, by definition,

$$\| g \|_{L^p(B_K^{(2)})} \leq \mathcal{D}_{p,2}(K^{-1}) \| g \|_{L^p(K^{-1}_w(B_K^{(2)}))}$$  

whenever $g \in \mathcal{S}(\mathbb{R}^2)$ has Fourier support in $\mathcal{N}_{K^{-1}}^{(2)}(P^1)$; here, for clarity, the bracketed exponents denote the dimension of the space in which a given ball or neighbourhood lies. Our first step is to demonstrate that (10) implies a decoupling inequality for the regions $U$. In particular, we will show

$$\| G \|_{L^p(B_K)} \leq \mathcal{D}_{p,2}(K^{-1}) \left( \sum_U \| G_U \|_{L^p(B_K)}^2 \right)^{1/2}$$  

(11)

whenever $G \in \mathcal{S}(\mathbb{R}^3)$ has Fourier support in $\mathcal{N}_{K^{-1}}^{(3)}(P^2) \cap \mathcal{N}_{K^{-1}}^{(3)}(P^1)$, where we identify $P^1 = \pi^{-1}(\ell(B_K)) \cap P^2$. Here $G_U$ is the Fourier restriction of $G$ to a subset of $U$ which is defined so as to satisfy $G = \sum_U G_U$.

To prove this claim, fix such an $G$ and define

$$G_x(y, z) := G(x, y, z).$$

It follows that $\hat{G}_x$ is supported in $\mathcal{N}_{O(K^{-1})}^{(2)}(P^1)$: indeed, for $(\xi, \eta) \in \mathbb{R} \times \mathbb{R}^2$ one has

$$\hat{G}(\xi, \eta) = \int e^{-2\pi i \xi \cdot \eta} \hat{G}_x(\eta) d\eta;$$

if $\eta \notin \mathcal{N}_{O(K^{-1})}^{(2)}(P^1)$, then the left-hand side of the above expression is identically 0 for all $\xi$ and therefore $\hat{G}_x(\eta) = 0$. Owing to this Fourier support condition, we can apply (10) to each of the $G_x$. Furthermore, if the $U$ are chosen subordinated to the $K^{-1/2}$-slabs appearing in the decoupling norm, then it follows that

$$\| G_x \|_{L^p(B_K^{(2)})} \leq \mathcal{D}_{p,2}(K^{-1}) \left( \sum_U \| G_{x,U} \|_{L^p(B_K^{(2)})}^2 \right)^{1/2},$$  

(12)

where $U^{(2)} := U \cap \{0, \eta : \eta \in \mathbb{R}^2\}$.

Notice, by the theory of distributions, we have that

$$G_{x,U^{(2)}}(y, z) = \int_{U^{(2)}} \hat{G}_x(\eta) e^{2\pi i \eta \cdot (y, z)} d\eta$$

$$- \int_{U^{(2)}} \delta(w - x) \hat{G}_w(\eta) e^{2\pi i \eta \cdot (y, z)} d\eta$$

$$- \int_{U^{(2)}} \int_{\mathbb{R}} e^{2\pi i \xi \cdot (w - x)} \hat{G}_w(\eta) e^{2\pi i \eta \cdot (y, z)} d\eta d\xi$$

$$- \int_{U^{(2)}} e^{2\pi i (\xi, \eta) \cdot (x, y, z)} \hat{G}(\xi, \eta) d\xi d\eta$$

$$- G_{U,x}(y, z).$$

Consequently, raising (12) to the $p$th power and integrating in $x$ over the interval $[a - cK, a + cK]$ (which corresponds to the projection of $B_K$ onto the $x$-axis), it
follows that
\[
\|G\|_{L^p(B_K)}^p \leq \mathcal{D}_{p,2}(K^{-1}) \int_{-\epsilon K}^{\epsilon K} \left( \sum_{U} \|G_{U,x}\|_{L^p(w_{B_K})}^2 \right)^{p/2} \, dx
\]
\[
\leq \mathcal{D}_{p,2}(K^{-1}) \left[ \sum_{U} \left( \int_{-\epsilon K}^{\epsilon K} \|G_{U,x}\|_{L^p(w_{B_K})}^p \, dx \right)^{2/p} \right]^{p/2}
\]
\[
\leq \mathcal{D}_{p,2}(K^{-1}) \left( \sum_{U} \|G_{U,x}\|_{L^p(w_{B_K})}^2 \right)^{p/2}.
\]

This proves the claim.

Returning to (9), the a priori estimate (11) implies the bound
\[
\left\| \sum_{\tau:K^{-1/2} \text{-reg.}} f_{\tau} \right\|_{L^p(B_K)} \leq \mathcal{D}_{p,2}(K^{-1}) \left( \sum_{U} \|f_U\|_{L^p(w_{B_K})}^2 \right)^{1/2}.
\]

At this point we’d like to get rid of the $U$ since they are not regions in the desired sense. We let $\{ \alpha \}$ denote a system finitely-overlapping $K^{-1/2}$-regions of $\mathcal{N}_{K^{-1}}(P^2)$ to which the $U$ are subordinated. Our task is now to substitute them with the $K^{-1/2}$-regions $\alpha$ they intersect with.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The projections of some U and $K^{-1/2}$-region $\alpha$.}
\end{figure}

Since for the $\tau$ subsuming $U$ one may suppose
\[ f_{\tau} = f_U + \sum_{\tau:K^{-1/2} \text{-reg.}} f_{\tau}, \]

Thus, for $x \in B_K \setminus E_{\text{trans}}$ it follows that
\[ |f_U(x)| \leq |f_{\alpha}(x)| + c_*(x) \]

where we have used the fact that the regions $\alpha$ which lie away from $\mathcal{L}(B_K)$ can contribute at most $K^{-2}c_*(x)$. Thus,
\[ \left( \sum_{U} \|f_U\|_{L^p(w_{B_K})}^2 \right)^{1/2} \leq \left( \sum_{\alpha:K^{-1/2} \text{-reg.}} \|f_{\alpha}\|_{L^p(w_{B_K})}^2 \right)^{1/2} + K^{-1} \left( \sum_{\tau:K^{-1/2} \text{-reg.}} \|f_{\tau}\|_{L^p(w_{B_K})}^2 \right)^{1/2}, \]

since there we are summing over $O(K^{1/2})$ ‘regions’ $U$. For the latter term we majorize the factor $K^{-b/3}$ by 1 and apply the inequality
\[ \|c_*\|_{L^p(w_{B_K})} \leq \left( \sum_{\tau:K^{-1} \text{-reg.}} \|f_{\tau}\|_{L^p(w_{B_K})}^2 \right)^{1/2}. \]
Letting $K$ applying parabolic rescaling, it follows that

$$\|f\|_{L^p(B_K)} \leq \mathcal{D}_{p,2}(K) \left[ \left( \sum_{\alpha, K^{-1/2} \text{-reg.}} \|f_{\alpha}\|_{L^p(w_{B_K})} \right)^{1/2} + \left( \sum_{\tau, K^{-1} \text{-reg.}} \|f_r\|_{L^p(w_{B_K})} \right)^{1/2} \right].$$

At this point we raise everything to the $p$-power and sum these inequalities over the $B_K$ to obtain an estimate over the whole $B_R \setminus E_{\text{trans}}$. Note that, by Minkowski’s inequality,

$$\sum_{B_K} \left( \sum_{\alpha, K^{-1/2} \text{-reg.}} \|f_{\alpha}\|_{L^p(w_{B_K})} \right)^{p/2} \leq \left( \sum_{\alpha, K^{-1/2} \text{-reg.}} \sum_{B_K} \|f_{\alpha}\|_{L^p(w_{B_K})} \right)^{2/p},$$

whilst

$$\sum_{B_K} \|f_r\|_{L^p(w_{B_K})}^p - \|f_r\|_{L^p(w_{B_K})}^p,$$

since $\sum_{B_K} w_{B_K}$ is a $w_{B_R}$-weight. We apply a similar analysis to the sum over $\tau$ and thereby deduce that

$$\|f\|_{L^p(B_R \setminus E_{\text{trans}})} \leq \mathcal{D}_{p,2}(K) \left[ \left( \sum_{\alpha, K^{-1/2} \text{-reg.}} \|f_{\alpha}\|_{L^p(w_{B_R})} \right)^{1/2} + \left( \sum_{\tau, K^{-1}} \|f_r\|_{L^p(w_{B_R})} \right)^{1/2} \right].$$

By the inductive hypothesis of the proposition, $\mathcal{D}_{p,2}(K) \leq C_\delta K^\delta$ and therefore, applying parabolic rescaling, it follows that

$$\|f\|_{L^p(B_R \setminus E_{\text{trans}})} \leq C_\delta K^\delta (\mathcal{D}_{p,3}(R/K) + \mathcal{D}_{p,3}(R/K^2)) \|f\|_{L^{p,n^{-1}}(w_{B_R})}$$

for all $0 < \delta \ll 1$. This controls $L^p$-norm over the non-transverse points.

### 3.4. Concluding the argument.

Combining our analysis of the transversal and co-planar cases, it follows that

$$\|f\|_{L^p(B_R)} \leq C_\delta K^\delta (\mathcal{D}_{p,3}(R/K) + \mathcal{D}_{p,3}(R/K^2) + K^{10} \mathcal{M}_{p,3}(R^{-1}, K^{-2})) \|f\|_{L^{p,n^{-1}}(w_{B_R})}$$

so that

$$\mathcal{D}_{p,n}(R) \leq C_\delta K^\delta (\mathcal{D}_{p,3}(R/K) + \mathcal{D}_{p,3}(R/K^2) + K^{10} \mathcal{M}_{p,3}(R^{-1}, K^{-2})).$$

Similar to the $n-2$ case, one now iterates this recurrence relation $m \sim \log_K R$ times and applies the trivial decoupling inequality at the resulting small scales. Consequently, one deduces that

$$\mathcal{D}_{p,n}(R) \leq K^{O(1)} R^{D(\delta + \log_K C)} \mathcal{M}_{p,n}(R, K^{-2}).$$

Letting $K - \omega^{-1/2}$ for some $R^{-1} \ll \omega \ll 1$, given $\epsilon > 0$ one may choose $\delta$ so that, if $\omega$ is sufficiently small depending only on $\epsilon$, then

$$\mathcal{D}_{p,n}(R) \leq \omega^{-O(1)} R^{\epsilon} \mathcal{M}_{p,n}(R, \omega),$$

as required. This concludes the proof of the proposition.

### 3.5. The proof of the parabolic rescaling lemma.

**Proof (of Lemma 8).** One may easily reduce the situation to the case where the Fourier support of $f$ lies on a $\rho$-region $\tau$ centred at the origin. Define the transformation

$$A : \mathbb{R}^n \to \mathbb{R}^n,$$

$$A(\xi_1, \ldots, \xi_{n-1}, \xi_n) - (\rho^{-1/2} \xi_1, \ldots, \rho^{-1/2} \xi_{n-1}, \rho^{-1} \xi_n);$$

the image $Ar$ of $\tau$ under $A$ is a $(\rho R)^{-1}$ neighbourhood of $P^{n-1}$. Moreover the $R^{-1/2}$-slabs contained in $\tau$ are mapped to all the $(\rho R)^{-1/2}$ slabs which cover $N_{(\rho R)^{-1}} P^{n-1}$. 

Thus, combining these observations and recalling the additional contribution (8), we see that so far we have managed to bound
Define $G := g \circ T$ so that $\text{supp} \hat{G} \subset N_{(\rho R)^{-1}}(P^{n-1})$. Therefore we can apply the decoupling estimate at scale $(\rho R)^{-1}$ to $G$, and obtain

$$
\|G\|_{L^p(B_R)} \leq \mathcal{D}_{p,n}(\rho R) \left( \sum_{\theta \in \mathcal{T}} \|g A_{\theta}\|_{L^p(w_{B_R})}^2 \right)^{1/2}.
$$

Now observe that $G_T = g \circ T$, so by a change of variables

$$
\|g\|_{L^p(A_{\rho R})} \leq \mathcal{D}_{p,n}(\rho R) \left( \sum_{\theta \in \mathcal{T}} \|g A_{\theta}\|_{L^p(w_{A_{\rho R}})}^2 \right)^{1/2}
$$

Now, $A_{\rho R}$ is an ellipsoid with axes $\rho^{1/2} R \times \cdots \times \rho^{1/2} R \times \rho R$ and $w_{B_R} \circ A^{-1}$ is a weight rapidly decaying away from $A_{\rho R}$. We can cover the ball $B_R$ with a finitely overlapping collection of ellipsoids of the form $A_{\rho R}$, and then raise the inequalities to the $p$th power and use Minkowski’s inequality to obtain the desired estimate.

\[\square\]

3.6. Closing remarks. The multi-linear $\ell^2$-decoupling estimate implies the multi-linear restriction estimate of [BCT06] in the partial range $2 \leq p \leq 2n/(n-1)$; this can be established via a very simple argument which is presented in the next set of notes. Therefore, by the proposition above one already has the linear decoupling estimate

$$
\|f\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p, n^{-1}(\mathbb{R}^n)}
$$

for the restricted range $2 \leq p \leq 2n/(n-1)$. These observations were made prior to Bourgain and Demeter’s work in [Bou13]. Thus the new contribution of [BD] was to devise the techniques needed to extend this estimate - or, rather, the multi-linear estimate - to include the missing range $\frac{2n}{n-1} < p$. These details are considered in the following lecture.

REFERENCES


Jonathan Hickman, Room 5409, James Clerk Maxwell Building, University of Edinburgh, Peter Guthrie Tait Road, Edinburgh, EH9 3FD.

E-mail address: j.e.hickman@sms.ed.ac.uk

Marco Vitturi, Room 4606, James Clerk Maxwell Building, University of Edinburgh, Peter Guthrie Tait Road, Edinburgh, EH9 3FD.

E-mail address: m.vitturi@sms.ed.ac.uk