LESSON 9: LIMITS AT INFINITY AND INFINITE LIMITS

Keywords

limit at infinity, infinite limit, vertical asymptote, horizontal asymptote

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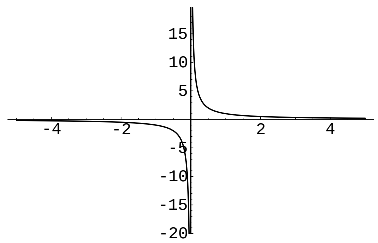
Let's look at the graph of $f(x) = \frac{1}{x}$. Notice that the values of $f(x)$ decrease more and more



$f(x) = \frac{1}{x}$

as x increases. Notice that if we plug-in some values, $f(2) = \frac{1}{2}$, $f(3) = \frac{1}{3}$, $f(100) = \frac{1}{100}$, and so on. We see that $f(x)$ gets smaller and smaller, approaching the value 0 as x gets larger and larger. In other words, we say that

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

Experimenting with large negative numbers would also lead us to write

$$\lim_{x\to-\infty} \frac{1}{x} = 0.$$

Does the "limit at infinity" always exist? Nope. Take our friendly sine function. The sine of any real number x lies between the values -1 and 1, and as x gets larger and larger, the values $\sin x$ will keep on oscillating between -1 and 1, therefore not approaching any specific number. Thus, the limit of $\sin x$ as x goes to infinity does not exist.

The rigorous (mathematical) definition of limits as x goes to $-\infty/\infty$ goes like this:

Let f be defined on $[c, \infty)$ for some number c. We say that $\lim\limits_{x\to\infty} f(x) = L$ if, for each $\epsilon > 0$ we can find a corresponding $M > 0$ such that if $x > M$, then $|f(x) - L| < \epsilon$.

In other words, we can make $f(x)$ as close to L as we wish, as long as we choose big enough values of x.

Let's see how the formal definition works to show that $\lim\limits_{x\to\infty} \frac{1}{x} = 0$:

Say we want $|f(x) - 0| < 0.0005$. If we choose x greater than 2000, we have $\frac{1}{x} < \frac{1}{2000} = 0.0005$. Hence, once $x > 2000$, all the $f(x)$ will be at a distance smaller than 0.0005 from 0, the limit of f as x goes to infinity.

Similarly, the formal definition for limit as x goes to negative infinity is as follows:

Let f be a function defined on $(-\infty, c]$, for some number c. We say that $\lim\limits_{x\to-\infty} f(x) = L$ if for each given $\epsilon > 0$ we can find a number $M < 0$ such that: if x is smaller than M, then $|f(x)-L| < \epsilon$.

Examples:

1. $\lim\limits_{x\to\infty} \frac{x}{1+x^2} = 0$.

   To see this, we can divide both the numerator and the denominator of the above fraction by x^2 to get:

$$\lim_{x\to\infty} \frac{x}{1+x^2} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x^2}+1} = \frac{\lim\limits_{x\to\infty} \frac{1}{x}}{\lim\limits_{x\to\infty} \frac{1}{x^2} + \lim\limits_{x\to\infty} 1}.$$

We then notice that both $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to\infty} \frac{1}{x^2}$ are equal to zero (why?) and therefore, using the limit theorems from previous lesson, we conclude that $\lim_{x\to\infty} \frac{x}{1+x^2} = 0$.

2. $\lim_{x\to-\infty} \frac{x}{1+x^2} = 0$.

   This follows from a similar computation to the above.

3. $\lim_{x\to\infty} \frac{x^3}{2x^3-100x^2} = \frac{1}{2}$.

   To compute this limit, we can divide both numerator and denominator of the fraction by x^3. Notice that for this division we need to remember that x cannot be zero, since we are making a division by x^3. However, since we are interested in the limit as x goes to infinity, we can consider that x is already larger than 0, so our division by x^3 is allowed. After the division we get:

$$\lim_{x\to\infty} \frac{x^3}{2x^3 - 100x^2} = \lim_{x\to\infty} \frac{1}{2 - 100\frac{1}{x}} = \frac{1}{2 - 100 \lim_{x\to\infty} \frac{1}{x}} = \frac{1}{2-0} = \frac{1}{2}.$$

Notice that in Examples 1 and 3 above, to compute the limit, we divided the numerator and the denominator by the "highest power of x" from the denominator. In the case of Example 1, the highest monomial in the denominator was x^2, and in Example 3, the highest monomial in the denominator was x^3. This trick of dividing by the highest power of the denominator will usually work. Later in this lesson we will see how to deal with the limits of fractions whose numerator actually has a higher degree polynomial than the denominator (which was not the case of the above three examples).

Up to now we have only seen cases where the limit is finite or does not exist. Sometimes, though, we can have infinite limits. For example, consider $\lim_{x\to 0+} \frac{1}{x}$. Notice that, as x approaches 0 from the right (that is, through the positive numbers), the fraction $\frac{1}{x}$ increases. For example, $f(0.01) = 100$, $f(0.00001) = 100000$, and so on. Looking at the graph of $f(x) = \frac{1}{x}$, we see that it "shoots up" when x gets near 0 from the right. In this case, we write:

$$\lim_{x\to 0+} \frac{1}{x} = \infty.$$

Examples:

1. $\lim_{x\to 2+} \frac{x}{x-2} = \infty$.

   Notice that as $x \to 2^+$, the numerator of the fraction goes to 2, while the denominator approaches 0 from the right. This makes the fraction bigger and bigger, and hence, in the limit, we say that $\lim_{x\to 2+} \frac{x}{x-2} = \infty$.

2. $\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$.

   Here, as $x \to 1$, $(x-1)^2$ is always a positive number (because it is a square), but with value approaching zero. Since $(x-1)^2$ is in the denominator, the fraction itself gets bigger and bigger. Hence $\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$.

The proper (mathematical) definition of infinite limit (when the limit is taken from the right hand side) is:

We say that $\lim_{x\to c^+} f(x) = \infty$ if for each positive number M we can find a corresponding $\delta > 0$ such that, if $0 < x - c < \delta$, then $f(x) > M$.

In other words, numbers that are near and greater than c should have larger and larger values of f the closer we get to c.

Of course, with some adjustments, we can define $\lim_{x\to\infty} f(x) = L$. The formal definition for this would be:

We say that $\lim_{x\to\infty} f(x) = \infty$ if, for any given positive number M, we can find a positive number N such that, if $x > N$, then $f(x) > M$.

One simple example is

$$\lim_{x\to\infty} x = \infty.$$

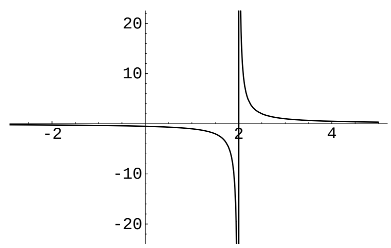
Another example is $\lim_{x\to\infty} \frac{x^2}{x-1} = \infty$.

When we divide numerator and denominator by the higher power of x in the denominator, that is, x, we get

$$\lim_{x\to\infty} \frac{x^2}{x-1} = \lim_{x\to\infty} \frac{x}{1-\frac{1}{x}} = \frac{\lim_{x\to\infty} x}{1-\lim_{x\to\infty} \frac{1}{x}} = \frac{\lim_{x\to\infty} x}{1-0} = \lim_{x\to\infty} x = \infty.$$

One thing that we should be careful about is limits like $\lim_{x\to 2} \frac{1}{x-2}$. Notice that $\lim_{x\to 2^-} \frac{1}{x-2} = -\infty$, while $\lim_{x\to 2^+} \frac{1}{x-2} = \infty$. Since the left hand limit and the right hand limit are not the same, the limit as x goes to 2 of $\frac{1}{x-2}$ does not exist!

Let's look at the graph of $f(x) = \frac{1}{x-2}$: Notice that there is a "invisible" vertical line at $x = 2$



that the graph is approaching when the x values are close to 2. The line $x = 2$ is called a *vertical asymptote.*

A line $x = c$ is *vertical asymptote* of the graph of $y = f(x)$ if any of the following four statements holds:

1. $\lim_{x\to c^+} f(x) = \infty$, or
2. $\lim_{x\to c^-} f(x) = \infty$, or
3. $\lim_{x\to c^+} f(x) = -\infty$, or
4. $\lim_{x\to c^-} f(x) = -\infty$.

Example:

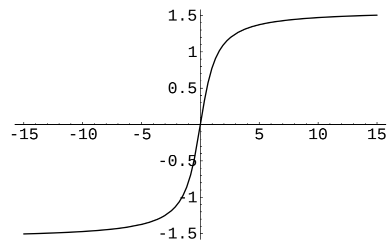
The line $x = 1$ is a vertical asymptote of $f(x) = \frac{2x}{x-1}$, because $\lim_{x\to 1^+} \frac{2x}{x-1} = \infty$.

In general, we pay attention to the lines $x = c$ where c is a value that annuls the denominator of our function f. Usually, there is a great chance that $x = c$ will be a vertical asymptote in that case, but this is not a general rule.

Notice that in the above example we have $\lim_{x\to\infty} \frac{2x}{x-1} = 2$. The line $y = 2$ is called a *horizontal asymptote* of the graph $y = f(x)$. If we look at the graph of $\frac{2x}{x-1}$, we see that for large positive values (in this case, also for large negative values) of x the graph of f tends to approach the line $y = 2$.

A line $y = b$ is called a *horizontal asymptote* of the graph $y = f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$.

Do not assume that the graph of a function can have only one horizontal asymptote. It may have two, as shown in the following picture:



Exercises

1. $\lim_{x\to\infty} \frac{x}{x-5} =$

   (a) 1 (b) -5 (c) 1 (d) does not exist

2. $\lim_{x\to\infty} \frac{x^2}{2-x^3} =$

   (a) 0 (b) 1 (c) 2 (d) does not exist

3. $\lim_{x\to\infty} \frac{x^4-1}{5x^4-2x^3+1} =$

   (a) 5 (b) $\frac{1}{5}$ (c) -1 (d) -2

4. $\lim_{x\to 1} \frac{3}{(x-1)^2} =$

   (a) 3 (b) 2 (c) does not exist (d) ∞

5. $\lim_{x\to 1} \frac{3}{x-1} =$

   (a) 3 (b) $-\infty$ (c) does not exist (d) ∞

6. $\lim_{t\to\infty} \frac{\sin^2 t}{t-5} =$

   (a) does not exist (b) ∞ (c) $-\frac{1}{5}$ (d) 0

7. $\lim_{x\to 3^-} \frac{x^2}{9-x^2} =$

   (a) ∞ (b) $-\infty$ (c) does not exist (d) $\frac{1}{9}$

Solutions

1(a) 2(a) 3(b) 4(d) 5(c) (because $\lim_{x\to 1^-} \frac{3}{x-1} = -\infty \neq \lim_{x\to 1^+} \frac{3}{x-1} = \infty$) 6(d) (because $|\sin^2 t| \leq 1$) 7(a) (because $\frac{x^2}{9-x^2} = \frac{x^2}{(3+x)(3-x)}$ and x^2, $3+x$ and $3-x$ are always positive numbers if x is smaller than 3. Since $x \to 3^-$, $3-x$ goes to zero and hence $\frac{3}{9-x^2}$ goes to infinity)

There is a shortcut for computing limits as x goes to infinity of rational functions: