LESSON 8: LIMITS INVOLVING TRIGONOMETRIC FUNCTIONS

Keywords
$\lim_{x\to 0}\frac{\sin x}{x}=1$, $\lim_{x\to 0}\frac{1-\cos x}{x}=0$

Basically, for a trigonometric function f, we will have $\lim_{x\to c} f(x) = f(c)$, as long as f is defined at c. In other words, in most cases we can just plug in the value of f at c to compute the limit.

For every real number c in the function's domain, we have:

1. $\lim_{x\to c}\sin x = \sin c$
2. $\lim_{x\to c}\cos x = \cos c$
3. $\lim_{x\to c}\tan x = \tan c$
4. $\lim_{x\to c}\cot x = \cot c$
5. $\lim_{x\to c}\sec x = \sec c$
6. $\lim_{x\to c}\csc x = \csc c$

Let's show why $\lim_{x\to c}\sin x = \sin c$. The proof goes through the following steps:

- Assume that $\lim_{x\to 0}\sin x = \sin 0 = 0$. (*we shall not prove this*)
- Then we show that $\lim_{x\to 0}\cos x = \cos 0 = 1$.

  This is not hard. Since $\cos x = \sqrt{1-\sin^2 x}$ for all real numbers x, we have $\lim_{x\to 0}\cos x = \lim_{x\to 0}\sqrt{1-\sin^2 x} = \sqrt{1-(\lim_{x\to 0}\sin x)^2} = \sqrt{1-0^2} = 1$.
- Now we use the previous two facts to get:

  $\lim_{x\to c}\sin c = \lim_{x\to c}\sin(c+0) = \sin c\cdot\cos 0 + \cos c\cdot\sin 0 = \sin c\cdot 1 + \cos c\cdot 0 = \sin c$

  which is what we wanted to show. (Recall that $\sin(a+b) = \sin a\cos b + \cos a\sin b$.)

Two Very Important Trigonometric Limits

We now introduce two very important limits:

1. $$\lim_{x\to 0}\frac{\sin x}{x} = 1$$
2. $$\lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

Again, we shall not prove how we get these two limits, but how to compute other related limits assuming these two.

Notice that if we try to plug in $x = 0$ directly in the above two limits, we get $\frac{0}{0}$, which is indeterminate. That's why they are important limits.

Examples:

1. $\lim_{x\to 0} \frac{\sin 3x}{x} = 3.$

   To compute this limit, we need to rewrite the original expression as

$$\lim_{x\to 0} 3\frac{\sin 3x}{3x} = 3\lim_{x\to 0}\frac{\sin 3x}{3x} = 3\cdot 1 = 3.$$

   We need to rewrite the expression $\lim_{x\to 0}\frac{\sin 3x}{x}$ into the form $\lim_{x\to 0} 3\frac{\sin 3x}{3x}$ because then we can substitute $y = 3x$ and get the limit $\lim_{y\to 0} 3\frac{\sin y}{y} = 3$, since y goes to 0 if x goes to 0. We cannot say that $\lim_{x\to 0}\frac{\sin 3x}{x} = 1$ because even though both $3x$ and x go to zero when x goes to 0, they don't go to zero *at the same speed*. In some sense, $3x$ goes to zero 3 times faster than x, and we can only use the first important limit $\lim_{x\to 0}\frac{\sin x}{x} = 1$ in this format, that is, what is inside the sine function needs to be exactly equal to the denominator, so that both things go to zero the same way. This will become clear in further examples.

2. $\lim_{x\to 0}\frac{\sin x}{5x} = \frac{1}{5}.$

   Again, because x and $5x$ don't go to 0 at the same rate, we need to rearrange the expression and then apply $\lim_{x\to 0}\frac{\sin x}{x} = 1$ wherever adequate. So

$$\lim_{x\to 0}\frac{\sin x}{5x} = \lim_{x\to 0}\frac{1}{5}\frac{\sin x}{x} = \frac{1}{5}\lim_{x\to 0}\frac{\sin x}{x} = \frac{1}{5}\cdot 1 = \frac{1}{5}.$$

3. $\lim_{x\to 0}\frac{\sin x^2}{x} = 0.$

   To compute this limit, we have to do:

$$\lim_{x\to 0}\frac{\sin x^2}{x} = \lim_{x\to 0}\frac{x}{x}\cdot\frac{\sin x^2}{x} = \lim_{x\to 0} x\frac{\sin x^2}{x^2} = \lim_{x\to 0} x\cdot\lim_{x\to 0}\frac{\sin x^2}{x^2} = 0\cdot 1 = 0.$$

4. $\lim_{t\to 1}\frac{\sin(t^2-1)}{t^2-1} = 1.$

   Here we notice that what's been plugged inside the sine function is $t^2 - 1$, the same as the denominator of the fraction, and $t^2 - 1$ goes to 0 when t goes to 1. Therefore, if we call $x = t^2 - 1$, the above limit becomes

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

5. $\lim_{x\to 0}\frac{1-\cos x}{\sin x} = 0.$

   This example is slightly different from the previous ones. The "trick" here is to divide both numerator and denominator by x. We get

$$\lim_{x\to 0}\frac{1-\cos x}{\sin x} = \lim_{x\to 0}\frac{\frac{1-\cos x}{x}}{\frac{\sin x}{x}} = \frac{\lim_{x\to 0}\frac{1-\cos x}{x}}{\lim_{x\to 0}\frac{\sin x}{x}} = \frac{0}{1} = 0.$$

6. $\lim_{x\to 0}\frac{\sin 2x}{\tan x} = 4.$

   For this problem, we will want to write $\tan x$ as $\frac{\sin x}{\cos x}$ first, and then do the adjustments we've been doing for the previous examples:

$$\lim_{x\to 0}\frac{\sin 2x}{\tan x} = \lim_{x\to 0}\frac{\sin 2x}{\frac{\sin x}{\cos x}}$$

now divide both numerator and denominator by $2x$ and get

$$\lim_{x\to 0}\frac{\frac{\sin 2x}{2x}}{\frac{\sin x}{2x\cos x}}=\frac{\lim_{x\to 0}\frac{\sin 2x}{2x}}{\lim_{x\to 0}\frac{\sin x}{2x\cos x}}=\frac{1}{\lim_{x\to 0}\frac{\sin x}{2x\cos x}}.$$

Now,

$$\lim_{x\to 0}\frac{\sin x}{2x\cos x}=\frac{1}{2}\lim_{x\to 0}\frac{\frac{\sin x}{x}}{\cos x}=\frac{1}{2}\frac{\lim_{x\to 0}\frac{\sin x}{x}}{\lim_{x\to 0}\cos x}=\frac{1}{2}\cdot\frac{1}{1}=\frac{1}{2}.$$

Hence $\lim_{x\to 0}\frac{\sin 2x}{\tan x}=\frac{1}{\frac{1}{2}}=2.$

Exercises

1. $\lim_{\theta\to\frac{\pi}{2}}\theta\cos(\theta)=$

   (a) $\frac{\pi}{2}$ (b) $-\frac{\pi}{2}$ (c) -1 (d) 0

2. $\lim_{x\to\pi}\frac{\sin x}{x}=$

   (a) 1 (b) 0 (c) -1 (d) $-\frac{1}{\pi}$

3. $\lim_{x\to 0}\frac{\sin x}{2x}=$

   (a) 2 (b) $\frac{1}{2}$ (c) 0 (d) 1

4. $\lim_{x\to 1}\frac{\sin(x^2-1)}{x-1}=$

   (a) 0 (b) 1 (c) 2 (d) -1

5. $\lim_{x\to 0}\frac{\tan 3x}{2x}=$

   (a) $\frac{3}{2}$ (b) $\frac{1}{2}$ (c) 3 (d) $\frac{1}{3}$

6. $\lim_{t\to 0}\frac{\tan^2 3t}{2t}=$

   (a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) 0 (d) $\frac{1}{2}$

7. $\lim_{t\to 0}\frac{\tan 5t}{\sin 2t}=$

   (a) 0 (b) 5 (c) $\frac{2}{5}$ (d) $\frac{5}{2}$

8. $\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=$

   (a) π (b) 0 (c) 1 (d) -1

9. $\lim_{x\to 1}\frac{1-\cos(x^2-1)}{x-1}=$

   (a) 0 (b) -1 (c) 2 (d) does not exist

10. $\lim_{x\to 0}\frac{\sin(3x)+4x}{x\sec x}=$

    (a) 7 (b) 0 (c) 3 (d) -1

11. $\lim_{t\to 0}\frac{\sin^2 3t}{2t}=$

    (a) $\frac{2}{3}$ (b) 2 (c) 0 (d) $\frac{1}{3}$

Solutions

1(d) 2(b) 3(b) 4(c) 5(a) 6(c) 7(d) 8(b) 9(a) 10(a) 11(c)