LESSON 7: LIMIT THEOREMS

Keywords
Substitution Theorem, Squeeze Theorem

The precise definition (the mathematical one) of limit goes like this:

To say that $\lim_{x \to c} f(x) = L$ *means that, given any positive number* ϵ*, no matter how small, there is a corresponding positive number* δ *such that* $|f(x) - L|$ *will be smaller than* ϵ *provided that* x *satisfies* $|x - c| < \delta$*.*

Basically, this is a way to formalize the fact that, one can always find an interval cointaining c such that every point different from c itself in that interval has its "f-value" as near as we wish from L.

We shall not work with the formal definition of limit, the intuitive one from the previous lesson will serve our purposes.

To make computation of limits easier, there are a few theorems we can use:

1. Let k be a constant (that is, a fixed real number). Then $\lim_{x \to c} k = k$.

   This basically states that, if f is a function that takes every real number to the value k, then the limit of f at every number is k.

   Example: $\lim_{x \to 2} 5 = 5$

2. $\lim_{x \to c} x = c$.

   This is again easy to see. Let $f(x) = x$. If x_1 is a point near c, then $f(x_1) = x_1$ is a point near c. Hence, in the limit, we have the above.

   Example: $\lim_{x \to 2} x = 2$.

3. Let k be a constant. Then $\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x)$.

   In other words, the limit of a number times a function is the same as the number times the limit of the function. We can "extract" the constant k from inside the limit.

   Example: $\lim_{x \to 2} 5x = 5 \lim_{x \to 2} x = 5(2) = 10$.

4. $\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$.

   In other words, the limit of the sum is the sum of the limits, *provided that all three limits above exist and are finite.*

   Example: $\lim_{x \to 2} x + 3 = \lim_{x \to 2} x \lim_{x \to 2} 3 = 2 + 3 = 5$.

5. $\lim_{x \to c} [f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$.

   This can be seen as an application of items 3 and 4 from above.

   Example: $\lim_{x \to 2} x - 3 = \lim_{x \to 2} x - \lim_{x \to 2} 3 = 2 - 3 = -1$.

6. $\lim_{x \to c} f(x)g(x) = \lim_{x \to c} f(x) \lim_{x \to c} g(x)$.

   In other words, the limit of the product is the product of the limit.

   Example: $\lim_{x \to 5} x^2 = \lim_{x \to 5} x(x) = (\lim_{x \to 5} x)(\lim_{x \to 5} x) = 5(5) = 25$.

7. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided the denominator does not vanish, that is, $\lim_{x\to c} g(x) \neq 0$.

   Example: $\lim_{x\to 5} \frac{x}{x-3} = \frac{\lim_{x\to 5} x}{\lim_{x\to 5} x-3} = \frac{5}{2}$.

8. (Substitution theorem) If f is a polynomial function or a rational function, then

$$\lim_{x\to c} f(x) = f(c)$$

   provided $f(c)$ is defined. In the case of rational functions, this means "provided the denominator at c is not zero". (Note: A polynomial function is a function of the form $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where the a_n's are real numbers. Examples: $x^3 - 5x + \pi$, $2x - 7$. Rational functions are quotients of two polynomial functions, for example: $\frac{x-1}{3x^4-1}$.)

   Example: $\lim_{x\to 1} \frac{x^3-4x^2+5x-1}{x+1} = \frac{1-4+5-1}{2} = \frac{1}{2}$.

9. (Squeeze theorem) Let f, g, and h be functions such that $f(x) \leq g(x) \leq h(x)$ for all x near c, except possibly at c (when dealing with limits, we don't care what happens AT c itself). If $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$.

   In other words, if our limit is squeezed between two things that have the same limit, then our limit has to equal to the limit of those two things, because it is always in between and got "squeezed".

   Example (*come back after Lesson 9*): $\lim_{x\to\infty} \frac{\sin x}{x} = 0$.

   We notice that $-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$. In Lesson 9 we will see that $\lim_{x\to\infty} -\frac{1}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$. Hence, by the Squeeze theorem, $\lim_{x\to\infty} \frac{\sin x}{x} = 0$.

Exercises

1. $\lim_{x\to 1} (2x+1) =$

   (a) 2 (b) 1 (c) 3 (d) 0

2. $\lim_{x\to 2} (3x^2 - 5) =$

   (a) 7 (b) 12 (c) 1 (d) -1

3. $\lim_{x\to 1} \frac{x-1}{x^2-1} =$

   (a) 0 (b) does not exist (c) $\frac{1}{2}$ (d) 1

4. $\lim_{x\to 2} \frac{2x+1}{5-3x} =$

   (a) 5 (b) -5 (c) $\frac{1}{5}$ (d) $\frac{-1}{3}$

5. $\lim_{x\to 1} (2x^3 + 15)^{17} =$

   (a) 17^{17} (b) 17 (c) 15^{17} (d) 2^{17}

6. $\lim_{x\to 1} \frac{x^2+x-2}{x^2-1} =$

   (a) does not exist (b) -1 (c) $\frac{-1}{2}$ (d) $\frac{3}{2}$

7. $\lim_{x\to -2} \frac{x^2+7x+10}{x+2} =$

   (a) 7 (b) 3 (c) does not exist (d) 1

8. Find examples to show that if $\lim_{x\to c}[f(x)+g(x)]$ exists, this does not imply that either $\lim_{x\to c} f(x)$ or $\lim_{x\to c} g(x)$ exists.

9. Find examples to show that if $\lim_{x\to c}[f(x)g(x)]$ exists, this does not imply that either $\lim_{x\to c} f(x)$ or $\lim_{x\to c} g(x)$ exists.

Solutions

1(c) 2(a) 3(c) 4(b) 5(a) 6(d) 7(b)