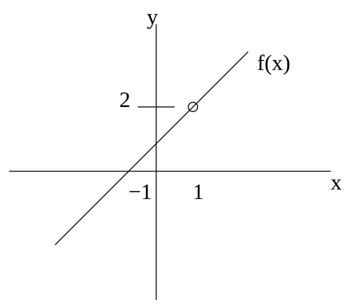
LESSON 6: INTRODUCTION TO LIMITS

Keywords
limit, right hand limit, left hand limit

The topics discussed so far are not really considered to be part of Calculus; they are just preparatory material. Calculus starts with the concept of *limit*. Before we start looking at the formal definition of limit, let's take a look at the function

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

This function is not defined at $x = 1$ because at that point the denominator vanishes. The graph of f looks like:



Notice that everywhere except at point $x = 1$ our f is given by

$$f(x) = \frac{(x+1)(x-1)}{x-1} = x + 1.$$

So what is the value of $f(x)$, when x is *approaching* 1? It is clear that when x is approaching 1, the values $f(x)$ are approaching 2. So, in this case, we are going to say that

$$\lim_{x \to 1} f(x) = 2,$$

or, in other words,

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

The intuitive definition of limit goes like this:

To say that $\lim_{x \to c} f(x) = L$ *means that when* x *is near but different from* c*, then* $f(x)$ *is near* L*.*

Notice that *we do not care what happens to* $f(x)$ *when* $x = c$*!* What concerns us is the behavior of the function f very, very near the point c but not really at c itself.

Here are some more examples:

1. Find $\lim_{x \to 2} (3x + 7)$.

   When x is near 2, $3x + 7$ is near $3(2) + 7 = 6 + 7 = 13$.

   Hence, $\lim_{x \to 2} (3x + 7) = 13$.

2. Find $\lim\limits_{x\to -3} \frac{x^2+x-6}{x+3}$.

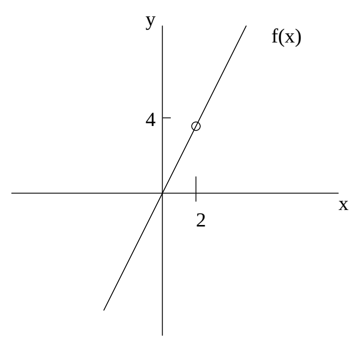
   Notice that $\frac{x^2+x-6}{x+3} = \frac{(x-2)(x+3)}{x+3}$, and we can cancel $x+3$ from the numerator with $x+3$ from the denominator *when* $x+3$ *is different from 0.* So, when x is near but different from -3, $\frac{(x-2)(x+3)}{x+3}$ is equal to $x-2$, which is near -5 when x is near -3.

We see that in the Example 1 we just "plugged in" the value of $x = 2$ inside the expression $3x + 7$ to find the limit. In Example 2 we needed to factor the terms first, do proper cancellations, and then plug in the value $x = -3$ in the expression $x - 2$. Do not think that computing limits is always as simple as that. For example, take

$$\lim_{x\to 0} \frac{\sin x}{x}.$$

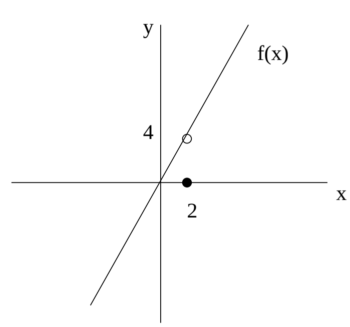
It is not clear how we can compute this limit, since when we plug in $x = 0$ we get $\frac{0}{0}$, which is undefined. It is also not clear how we can factor the trigonometric sine function to cancel with the x in the denominator. The above limit is actually equal to 1, but we shall deal with this limit in later lessons.

How do we look at the graph of $y = f(x)$ and decide what is $\lim\limits_{x\to c} f(x)$? For example, let the graph of $y = f(x)$ be as shown below. What is $\lim\limits_{x\to 2} f(x)$? Notice there is a "hole" in the



graph where x equals 2. But we see that, when x is near 2, the values of $f(x)$ are near 4. Hence $\lim\limits_{x\to 2} f(x) = 4$. It is as if we were "covering the hole" in the graph.

One important thing that should be emphasized is the fact that the above limit does not depend on the value of $f(2)$ itself. In the previous graph $f(2)$ was not defined, but if the graph of f looked like: $\lim\limits_{x\to 2} f(x)$ still equals 4, even though $f(2) = 0$.



Another important thing is the fact that limits sometimes might not exist. If the graph of f looked like Figure 1 then $\lim\limits_{x\to 2} f(x)$ does not exist. This is because when x is near but smaller than

2, then $f(x)$ is near the value 4. But if x is near but greater than 2, then $f(x)$ is near the value 6. Since these values don't agree with each other, $\lim_{x\to 2} f(x)$ does not exist.

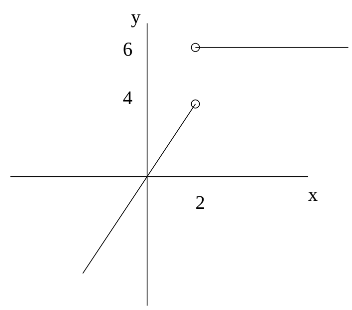


Figure 1

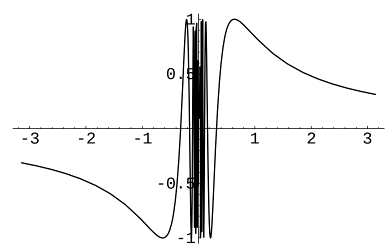
When the function takes a nice "jump" at some value $x = c$, we can still define some sort of limit, the *right hand limit* and the *left hand limit.*

To say that $\lim_{x\to c^+} f(x) = L$ means that when x is near but on the right of c (that is, x is greater than c), then $f(x)$ is near L. We call L the *right hand limit* pf f when x approaches c.

Similarly, to say that $\lim_{x\to c^-} f(x) = M$ means that when x is near but on the left of c (that is, x is smaller than c), then $f(x)$ is near M. We call M the *left hand limit* of f when x approaches c.

Of course, $\lim_{x\to c} f(x) = L$ if and only if both limits $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ exist and equal L.

Beware of even worse situations (that is, it is not even due to a "jump" of the function at a point) where limits don't exist. Take the function $g(x) = \sin(\frac{1}{x})$. This function has graph



The function g is not defined at $x = 0$, and we see that it is impossible to find $\lim_{x\to 0} g(x)$, since the graph of g oscillates more and more as x gets near 0, never approaching any fixed value. In this case, there are no left or right hand limits for g when x approaches 0!

Exercises

1. Find $\lim_{x\to 3}(x-3)$.

   (a) 6 (b) -3 (c) 0 (d) -6

2. Find $\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}$.

   (a) 0 (b) 2 (c) 1 (d) -1

3. Find $\lim_{t\to 7} \frac{\sqrt{(t-7)^3}}{t-7}$.

   (a) 0 (b) 1 (c) does not exist (d) -7

4. Find $\lim_{x \to 5} \frac{x^2-25}{x-5}$.

   (a) does not exist (b) 10 (c) 0 (d) -5

5. Find $\lim_{x \to 0} \frac{x^4+2x^3-x^2}{x^2}$.

   (a) 0 (b) 1 (c) -1 (d) does not exist

6. For the function f graphed in Figure 1, find $f(-3)$.

   (a) does not exist (b) 1 (c) 2 (d) 3

7. For the function f graphed in Figure 1, find $\lim_{x \to -3} f(x)$.

   (a) does not exist (b) 1 (c) 2 (d) 3

8. For the function f graphed in Figure 1, find $f(-2)$.

   (a) does not exist (b) 1 (c) 2 (d) 3

9. For the function f graphed in Figure 1, find $\lim_{x \to -2} f(x)$.

   (a) does not exist (b) 1 (c) 2 (d) 3

10. For the function f graphed in Figure 1, find $f(1)$.

    (a) does not exist (b) 1 (c) 2 (d) 3

11. For the function f graphed in Figure 1, find $\lim_{x \to 1} f(x)$.

    (a) does not exist (b) 1 (c) 2 (d) 3

12. For the function f graphed in Figure 1, find $\lim_{x \to 1^+} f(x)$.

    (a) does not exist (b) 1 (c) 2 (d) 3

13. For the function f graphed in Figure 1, find $\lim_{x \to 1^-} f(x)$.

    (a) does not exist (b) 1 (c) 2 (d) 3

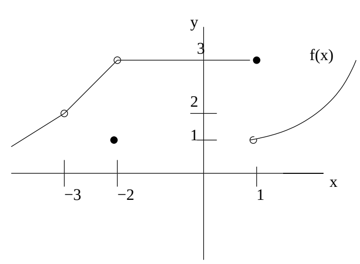


Figure 1

Solutions

1(c) 2(b) 3(a) 4(b) 5(c) 6(a) 7(c) 8(b) 9(d) 10(d) 11(a) 12(b) 13(d)