LESSON 5: THE TRIGONOMETRIC FUNCTIONS

Keywords
angles, radians, π, sine, cosine, tangent, cotangent, secant, cosecant, periodic function, period, amplitude

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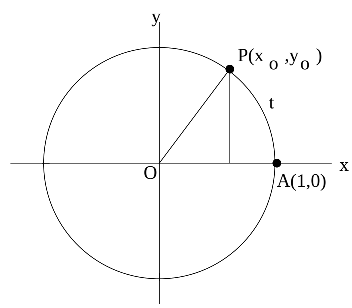
Here we define the trigonometric functions based on the unit circle. The unit circle C is the circle with radius 1 and center at the origin. It has as its equation $x^2 + y^2 = 1$ (notice that this is not a function of x nor of y). Let A be the point $(1, 0)$ and let t be a positive real number. There is a single point P on the unit circle C such that the distance of the arc AP, measured counterclockwise, is equal to t. This point P will have coordinates (x_0, y_0) on the circle C.

We define the *sine* of t as y_0, and the *cosine* of t as x_0.

If t is a negative real number, we start from point $A = (1, 0)$ and go along the unit circle C in the clockwise direction until a point P where the arc AP has length $|t|$. If the coordinates of the point P are (x_0, y_0), we define $\cos t = x_0$ and $\sin t = y_0$.



To make these definitions clear, let's see some examples. Remember that the circumference of a circle with radius 1 is equal to 2π.

Examples

1. If $t = \pi$, then $\cos t = -1$ and $\sin t = 0$.

   Since π is half the length of the circumference of C, the corresponding point P is exactly halfway aroung the circle from the point A, and it has coordinates $(-1, 0)$. Therefore, $\cos \pi = -1$ and $\sin \pi = 0$.

2. If $t = 2\pi$, then $\cos t = 1$ and $\sin t = 0$.

   We trace the circle in the counterclockwise direction. As the circumference of C equals 2π, the corresponding point P is equal to A, which has coordinates $(1, 0)$. Notice that if t is any integer multiple of 2π (either positive or negative), we have $\sin t = 0$ and $\cos t = 1$.

3. If $t = -\frac{\pi}{2}$, then $\cos t = 0$ and $\sin t = -1$.

   Since t is negative, we trace the circle C in the clockwise direction until we reach a point P such that the arc AP has length $\pi/2$. The length $\pi/2$ is one fourth the length of the circumference of C, 2π. Therefore, the point P is the southern most point of the circle, with coordinates $(0, -1)$. Notice that if $t = \frac{3\pi}{2}$, then $\cos t = 0$ and $\sin t = -1$ also.

Properties of Sine and Cosine

A number of facts follow directly from our definitions of the sine and cosine functions.

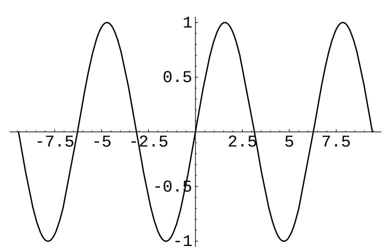
- Because the unit circle has circumference 2π, the values t and $t + 2\pi$ determine the *same point* P on the circle. Therefore, $\sin t = \sin(t + 2\pi)$ and $\cos t = \cos(t + 2\pi)$.

- The values t and $-t$ determine points P_1 and P_2 (respectively) on the circle that are symmetric with respect to the x-axis. Hence $\cos t = \cos(-t)$ and $\sin t = -\sin(-t)$. We conclude that cosine is an even function and sine is an odd function.
- The values t and $\pi - t$ determine points P_1 and P_2 that are symmetric with respect to the y-axis, hence $\sin t = \sin(\pi - t)$ and $\cos(t) = -\cos(\pi - t)$.
- The values t and $t + \pi$ determine points P_1 and P_2 that are symmetric with respect to the origin, therefore $\sin(t + \pi) = -\sin t$ and $\cos(t + \pi) = -\cos t$.
- The values t and $\pi/2 - t$ determine points P_1 and P_2 that are symmetric with respect to the line $y = x$ (it is easier if we try to picture this for $0 < t < \pi/2$). Hence the coordinates of P_1 and P_2 are "flipped", so: $\cos t = \sin(\pi/2 - t)$ and $\sin t = \cos(\pi/2 - t)$.
- Since for any value of t the corresponding point $P = (x_0, y_0)$ is on the unit circle, its coordinates must satisfy the equation $x^2 + y^2 = 1$. That is, $x_0^2 + y_0^2 = 1$ for any P. Hence $\sin^2 t + \cos^2 t = 1$ for any value of t.
- Clearly $-1 \leq \cos t \leq 1$ and $-1 \leq \sin t \leq 1$ for all t.

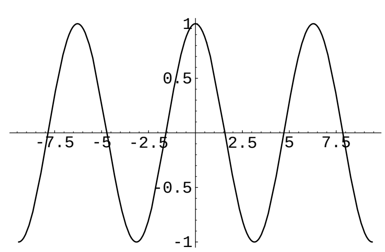
Other Trigonometric Functions

The other elementary trigonometric functions are *tangent, cotangent, secant* and *cosecant.* They are defined as follows.
$\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{\cos x}{\sin x}$, $\sec x = \frac{1}{\cos x}$ and $\csc x = \frac{1}{\sin x}$.
Notice that tangent and secant are not defined for $x = \frac{\pi}{2} + 2k\pi$, where k is an integer, and cotangent and cosecant are not defined for $x = 2k\pi$, where k is any integer, because those are the points where their denominators vanish.

Graphs of Sine and Cosine



The graph of the sine function



The graph of the cosine function

From the graph of sine and cosine we can derive the graphs of other elementary trigonometric functions.

Notice that both graphs repeat themselves on adjacent intervals of length 2π. This is due to the fact that $\sin(t + 2\pi) = \sin t$ and $\cos(t + 2\pi) = \cos t$. We say that sine and cosine are *periodic functions with period* 2π.

A function f is *periodic* if there is a number $p > 0$ such that

$$f(t + p) = f(t)$$

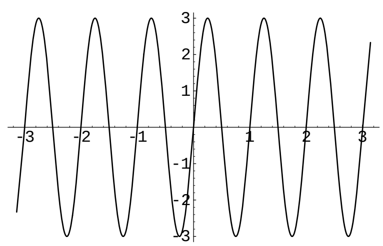
for all t in the domain of f.

The *period* of f is the smallest positive number that satisfies the above.

If the periodic function f attains a maximum and a minimum values, then the half-difference between the max and min values is called the *amplitude* of f.

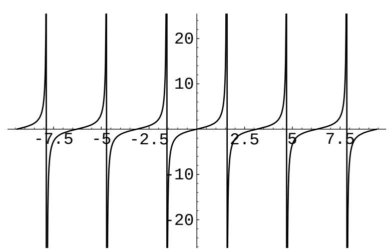
Examples:

1. The function $f(x) = 3\sin(2\pi x)$ has period 1 and amplitude 3.



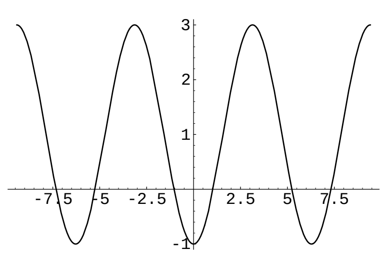
$f(x) = 3\sin(2\pi x)$

2. The function $g(x) = \tan x$ has period π and no defined amplitude (since it does not attain a maximum nor a minimum).



$g(x) = \tan x$

3. The function $h(x) = 2\cos(x - \pi) + 1$ has period 2π and amplitude 2.



$h(x) = 2\cos(x - \pi) + 1$

Relation between degrees and radians

In this lesson our angles were all measured in radians. A *radian* is by definition the angle corresponding to an arc of length 1 on the unit circle (since the unit circle has circumference 2π, a complete revolution corresponds to 2π radians). The conversion from degrees to radians and vice-versa is not hard, as long as we recall that $180^o = \pi$ radians. For example, the right angle is $\frac{\pi}{2}$ radians, which corresponds to 90 degrees. A circle has 2π radians, or 360 degrees.

Examples:

1. How many degrees is $\frac{5\pi}{6}$ radians?

   $\frac{5\pi}{6}$ radians is equal to $\frac{5(180)}{6} = \frac{900}{6} = 150$ degrees.

2. How many radians is 270^o?

$$\begin{array}{rcl} x & \rightarrow & 270^o \\ \pi & \rightarrow & 180^o \end{array}$$

   gives $\frac{x}{\pi} = \frac{270^o}{180^o} = \frac{3}{2}$. Hence $x = \frac{3\pi}{2}$ radians.

Trigonometric Indentities

There are many important trigonometric indentities that will come in useful later, in derivation and integration lessons. From the indentity $\sin^2 x + \cos^2 x = 1$ one can derive the identities $1 + \cot^2 x = \csc^2 x$ and $1 + \tan^2 x = \sec^2 x$.

We also have the addition identities (which we shall not prove)

$$\begin{array}{rcl} \sin(x+y) & = & \sin x \cdot \cos y + \cos x \cdot \sin y \\ \cos(x+y) & = & \cos x \cdot \cos y - \sin x \cdot \sin y \end{array}$$

from which we derive $\sin 2x = 2 \sin x \cos y$ and $\cos 2x = \cos^2 x - \sin^2 x$.

The addition identities will be used in Lesson 14.

In this study guide, we will always mention the trigonometric identities before using them.

Exercises

1. $\frac{7\pi}{6}$ radians is equivalent to x degrees, where x is

   (a) 105 (b) 420 (c) 210 (d) 195

2. 240^o is equivalent to w radians, where w is

   (a) $\frac{4}{3}\pi$ (b) $\frac{3}{4}\pi$ (c) $\frac{4}{3}$ (d) $\frac{5}{6}\pi$

3. Which of the following represent the same graph?

   (a) $y = \sin\left(x + \frac{\pi}{2}\right)$ (b) $y = -\cos(\pi - x)$ (c) $y = \cos(x - \pi)$ (d) $y = \cos\left(x - \frac{\pi}{2}\right)$

4. What are, respectively, the period and amplitude of the periodic function $f(t) = \frac{3}{5}\cos(\frac{\pi}{3}t) - 1$?

   (a) $\frac{5}{3}$ and $\frac{3}{\pi}$ (b) $\frac{5}{3}$ and $\frac{6}{\pi}$ (c) $\frac{3}{5}$ and $\frac{6}{\pi}$ (d) $\frac{3}{5}$ and 6

5. Which of the following are even functions?
   (a) $t \sin t$ (b) $\sin^2 t$ (c) $\csc t$ (d) $|\sin t|$ (e) $\sin(\cos t)$

Solutions

1(c) 2(a) 3(a-d) 4(d) 5(a-b-d-e)