

LESSON 4: THE INVERSE FUNCTION

Keywords

inverse function, f^{-1} , graph of the inverse function

Recall what a function is. It is basically a machine that takes one input and gives one single output for that input. Now we try to “invert” the machine. If we know a specific output that the machine gives, can we retrieve what the corresponding input was?

For example, say f is a function defined by $f(x) = 3x$. Then we know that f takes a number and multiplies it by 3. If a particular output of f was 45, we know that the input that f was given must have been 15, because 45 is three times 15. If the output of f was 4, then we know the input must have been $\frac{4}{3}$. So we can now construct an “inverting machine”: a function that, given the output of f as its input, will give us back whatever we plugged into f originally. In our particular example, we know that the function g defined by $g(x) = \frac{x}{3}$ is the inverting machine for f . We have $f(15) = 45$, so $g(45) = 15$. We have $f(\frac{4}{3}) = 4$, so $g(4) = \frac{4}{3}$. The function g is called the *inverse function* of f .

Is a function always invertible?

NO! What happened in our previous example that helped us determine the inverting machine g was the fact that two different inputs always give two different outputs, so when we see the number 45, we know exactly which number was plugged into f originally. There was no ambiguity. Does the expression “two different inputs always give two different outputs” sound familiar? That is our definition of a *injective function*, in our first lesson.

Let’s see an example where a function does not have an inverse because it is not injective. Look at our well-known function f that takes a name and outputs the first letter of the name. If g is the inverse function of f , what is $g(a)$? So $g(a)$ must be a name that starts with the letter a . But there are so many of them! We have *Andrea, Anna, Andy, Abe, Anderson*, and many more. Which one of them should be $g(a)$ we don’t know. Thus f does not have an inverse function.

Therefore, in order to be invertible, a function needs to be *injective*.

There is also another restriction for a function to be invertible. Let’s look at the following example: Say f is a function with domain the set of integers. That is, f only takes as input the integer numbers, like $-3, -1, 0, 4, 7, \dots$. Let $f(x) = 2x$. Hence f doubles an integer. Now, what would be the inverse function of f ? Of course, the inverse should take an integer and divide by 2, so if g was the inverse, it would be $g(x) = \frac{x}{2}$. Then what is $g(3)$? In our minds, $g(3)$ should be 1.5. But 1.5 is not an integer, so it would never be an input to our f . We figure that g can take only *even* integer numbers as input, because *the range of f only has even integers!*

So the second restriction (the first being that f needs to be injective) is: the inverse function of f should have as its domain the range of f . In other words, only possible outputs of f can work as inputs to the inverse of f .

Notation

The inverse function of f is usually denoted as f^{-1} .

Examples:

1. If $f(x) = 2x$, then $f^{-1}(x) = \frac{x}{2}$.
2. If $f(x) = x - 3$, then $f^{-1}(x) = x + 3$.
3. If $f(x) = x^3$, then $f^{-1}(x) = \sqrt[3]{x}$.
4. If $f(x) = \frac{x+4}{5}$, then $f^{-1}(x) = 5x - 4$.

Notice that in the first three examples it wasn't hard to guess what f^{-1} should be. But in Example 4, things start looking a bit more complicated.

There is a general technique on how to find the inverse of a function (though it might not always work, especially if the function we are considering is not really invertible!).

1. Let $f(x) = \frac{x+4}{5}$.

- Write it as $y = \frac{x+4}{5}$.
- Now try to write x as a function of y instead. That is, try to write the equation such that the only x symbol we have is on the left hand side, and by itself.

$$\begin{aligned} y &= \frac{x+4}{5} \\ 5y &= x+4 \\ 5y-4 &= x \\ x &= 5y-4 \end{aligned}$$

- Once we managed to write x as a function of y , switch the two of them and get our inverse function.

$$\begin{aligned} y &= 5x-4 \\ f^{-1}(x) &= 5x-4 \end{aligned}$$

2. Let $f(x) = \sqrt[3]{x-1} + 3$.

- Write it as $y = \sqrt[3]{x-1} + 3$.
- Now try to write x as a function of y instead.

$$\begin{aligned} y &= \sqrt[3]{x-1} + 3 \\ y-3 &= \sqrt[3]{x-1} \\ (y-3)^3 &= x-1 \\ (y-3)^3 + 1 &= x \\ x &= (y-3)^3 + 1 \end{aligned}$$

- Once we managed to write x as a function of y , switch the two of them and get our inverse function.

$$\begin{aligned} y &= (x-3)^3 + 1 \\ f^{-1}(x) &= (x-3)^3 + 1 \end{aligned}$$

There is a way to double-check your inverse function: the property

$$f^{-1}(f(x)) = x$$

should always be true.

In the Example 2 above, to check whether the function $f^{-1}(x) = (x-3)^3 + 1$ is actually the inverse of the function $f(x) = \sqrt[3]{x-1} + 3$, let's compute $f^{-1}(f(x))$:

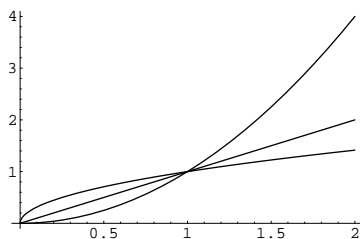
$$f^{-1}(f(x)) = f^{-1}(\sqrt[3]{x-1} + 3) = (\sqrt[3]{x-1} + 3 - 3)^3 + 1 = (\sqrt[3]{x-1})^3 + 1 = (x-1) + 1 = x.$$

So our checking showed that we were correct about the formula for f^{-1} .

Graph of the inverse function

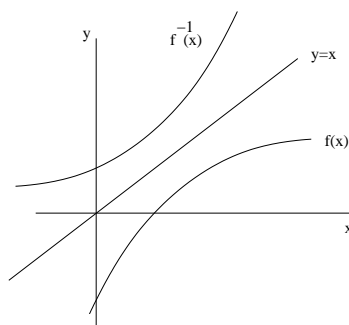
How does one draw the graph of f^{-1} given the graph of f ?

Let's see an example. Take $f(x) = x^2$, for $x > 0$. Then $f^{-1}(x) = \sqrt{x}$. Their graphs look like:



Notice that the two graphs are symmetric to each other with respect to the line $y = x$.

This is usually true. To draw the graph of f^{-1} , reflect the graph of f with respect to the line $y = x$.



Exercises

1. If $f(x) = \frac{4}{3}\sqrt{x-5}$ then $f^{-1}(x) =$

(d) $(\frac{3}{4}x)^2 + 5$

2. The graph of f is depicted in Figure 1. The graph of f^{-1} should look like:

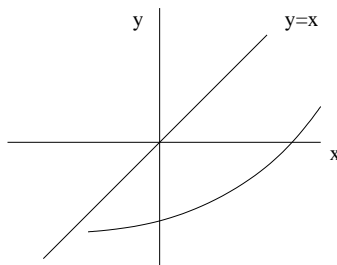
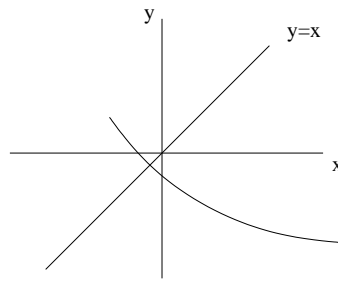
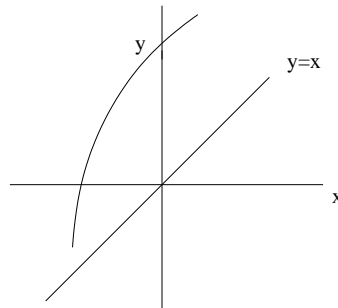


Figure 1

4



(a)



(b)

3. If $f^{-1}(x) = 2x$, then $f(x) =$
(a) $2x$ (b) $x + 2$ (c) $\frac{x}{2}$ (d) does not exist

Solutions

1(d) 2(b) 3(c)