LESSON 3: OPERATIONS ON FUNCTIONS

Keywords
composition of functions, translation of graphs

Given two functions f and g, they can be added to produce a new function $f+g$, defined by $(f+g)(x) = f(x) + g(x)$. Similarly, we can subtract function g from function f, and get a new function $f-g$ defined by $(f-g)(x) = f(x) - g(x)$; multiply function f by function g, obtaining function $(fg)(x) = f(x)g(x)$; divide function f by function g, provided $g(x)$ is never equal to 0, producing the function $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$.

Examples:

1. Let $f(x) = 2x$ and let $g(x) = x^2 + 1$. Then
   - $(f+g)(x) = f(x) + g(x) = 2x + x^2 + 1$
   - $(f-g)(x) = f(x) - g(x) = 2x - (x^2+1) = 2x - x^2 - 1$
   - $(fg)(x) = f(x)g(x) = (2x)(x^2+1) = 2x^3 + 2x$
   - $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x}{x^2+1}$ (notice that $g(x) \geq 1$ always, so it can be put in the denominator!)
   - $(g-f)(x) = g(x) - f(x) = x^2 + 1 - 2x$
2. Let $f(x) = x$ and let $g(x) = x - 3$. Then
   - $(f+g)(x) = x + x - 3 = 2x - 3$
   - $\left(\frac{f}{g}\right)(x) = \frac{x}{x-3}$ *but the domain should not include the point* $x = 3$, because $g(3) = 0$ and we cannot have zero as a denominator!

Composition of Functions

Given two functions f and g, we can also construct another function $f \circ g$, called the *composition* of f and g, such that $(f \circ g)(x) = f(g(x))$. What this definition means is: plug in a value x as the input to the function g. Whatever g outputs, we plug in as the input to the function f.

From the above you may notice that we might not always be able to compose two arbitrary functions. *We need the range of g to be inside the domain of f!* In other words, outputs of g have to be possible inputs of f for us to be able to construct $f \circ g$. For example, think of the function g that, given a letter from the English alphabet as input, outputs the next letter (we also define $g(z)$ to be equal to a). Then, $g(d) = e$, $g(j) = k$, and so on. Let the function f be such that, given an English name, it outputs the first letter of the name. For example, $f(Harry) = h$. Now, let's construct the composition of g with f. We have $(g \circ f)(Harry) = g(f(Harry)) = g(h) = i$. Similarly, $(g \circ f)(Andrea) = g(f(Andrea)) = g(a) = b$. So what the function $g \circ f$ does is, takes an English name as an input, and gives as corresponding output the letter that follows the first letter of the name.

What happens if we consider $f \circ g$? Notice that f takes as input only English names, while g gives as output only a letter. We do not have English names with just one letter "B". Therefore, composing f with g does not make sense.

Examples:

1. Let $f(x) = x^2 - 5$ and let $g(x) = x - 3$. Then $(f \circ g)(x) = f(x-3) = (x-3)^2 - 5 = x^2 - 6x + 4$ and $(g \circ f)(x) = g(x^2 - 5) = x^2 - 5 - 3 = x^2 - 8$. *Notice that, in general, $f \circ g$ and $g \circ f$ are two different functions!*

2. Let $f(x) = \sqrt{x}$ and $g(x) = \frac{x-3}{2}$. Then $(f \circ g)(x) = \sqrt{\frac{x-3}{2}}$. Since we do not have square root of negative numbers, we need $x - 3 \geq 0$ or $x \geq 3$. So the domain of $f \circ g$ consists of real numbers ≥ 3.

3. Let $f(x) = x^3 - 2$. Then $(f \circ f)(x) = f(f(x)) = (x^3 - 2)^3 - 2$.

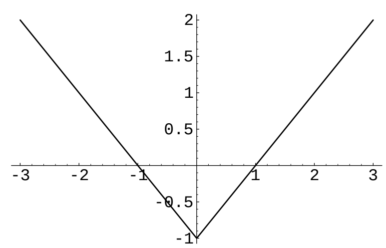
Translations

How are the graphs of

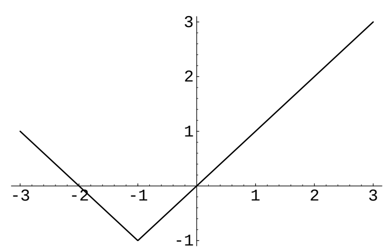
$$\begin{array}{cccc} y = f(x) & y = f(x+1) & y = f(x-1) & y = |f(x)| \\ y = f(x) + 1 & y = f(x) - 1 & y = -f(x) & \end{array}$$

related to each other?

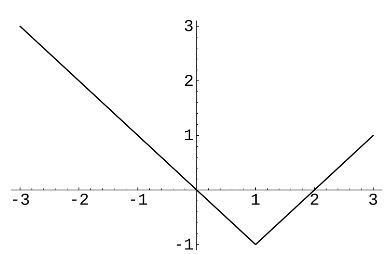
Shown below are the corresponding graphs for the case where $f(x) = |x| - 1$.



$f(x) = |x| - 1$

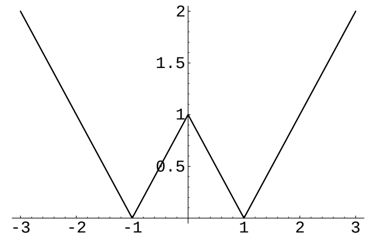


$f(x+1) = |x+1| - 1$

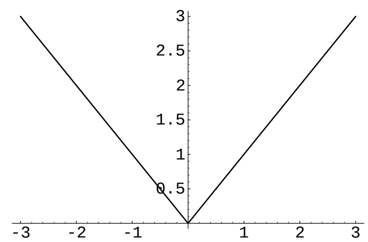


$f(x-1) = |x-1| - 1$

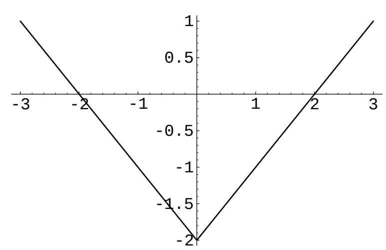
Notice that the graph of $y = f(x+1)$ is basically the graph of $y = f(x)$ *shifted* to the *left* by one unit. Notice also that the graph of $y = f(x-1)$ is the graph of $y = f(x)$ shifted to the *right* by one unit. This "shifting" is true for more cases. Let c be any positive real number. The graph of $y = f(x+c)$ is equal to the graph of $y = f(x)$ shifted left by c units, and the graph of $y = f(x-c)$ is equal to the graph of $y = f(x)$ shifted to the right by c units.



$|f(x)| = ||x| - 1|$

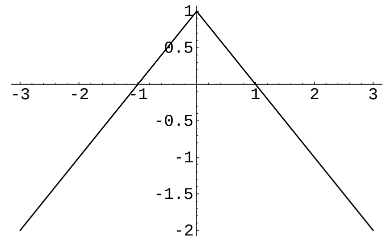


$f(x) + 1 = |x|$



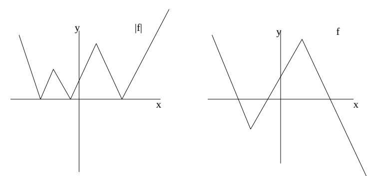
$f(x) - 1 = |x| - 2$

Notice now that the graph of $y = f(x) + 1$ is equal to the graph of $y = f(x)$ shifted *up* by one unit. Similarly, the graph of $y = f(x) - 1$ is the graph of $y = f(x)$ shifted *down* by one unit. This also works in general. The graph of $y = f(x) - 5$ is equal to the graph of $y = f(x)$ shifted down by 5 units, and so on.

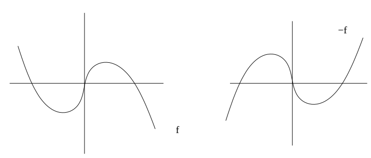


$-f(x) = 1 - |x|$

The graph of $y = |f(x)|$ can be drawn from the graph of $y = f(x)$ in the following manner: first draw the graph of $y = f(x)$. Wherever the graph is *below* the horizontal x-axis, we reflect it up, so that the resulting graph lies above the x-axis. (see figure below)



The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected everywhere with respect to the horizontal axis (see figure below).



Exercises

1. If $f(x) = x$, $g(x) = 2x^2 + 3$, then $(fg)(0) =$

   (a) 3 (b) 0 (c) 2 (d) 4

2. If $f(x) = \sqrt{x+3}$, $g(x) = x$, then $(f+g)(1) =$

   (a) $\sqrt{3}$ (b) 2 (c) 1 (d) 3

3. If $f(x) = x^2$, $g(x) = x - 3$, then $(f+g)(1) - (f-g)(1) =$

   (a) 2 (b) 4 (c) -2 (d) -4

4. If $f(x) = x^3$, $g(x) = x + 1$, then $\left(\frac{f}{g}\right)(-1) =$

   (a) -1 (b) $-\frac{1}{2}$ (c) does not exist (d) 0

5. If $f(x) = \sqrt[3]{x-3}$, and $g(y) = y^2$, then $(f \circ g)(x) =$

   (a) $\sqrt[3]{x^2 - 3}$ (b) $\sqrt[3]{y^2 - 3}$ (c) $\sqrt[3]{x-3}$ (d) $[\sqrt[3]{x-3}]^2$

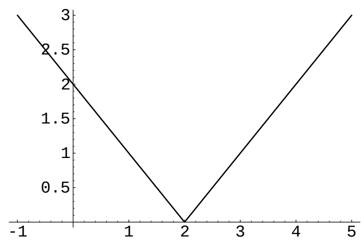
6. If $f(x) = \frac{x-1}{x^2}$ and $g(x) = 2$, then $(f \circ g)(0) =$

   (a) does not exist (b) $\frac{1}{4}$ (c) 2 (d) $\frac{-1}{2}$

7. If $f(x) = x^2 + 1$, then $(f \circ f \circ f)(x) =$

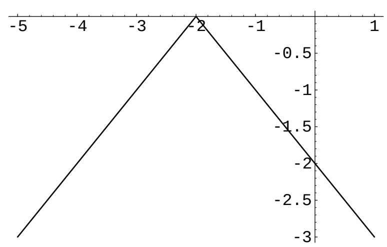
   (a) $x^2 + 1$ (b) $((x^2+1)^2+1)^2+1$ (c)$(x^2+1)^2+1$ (d) $((x+1)^2+1)^2+1$

8. If $f(x) = |x|$, then the graph of $y = f(x + 2)$ looks like:
(a) True (b) False



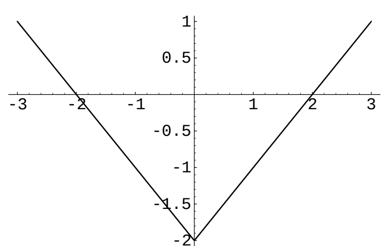
Exercise 8

9. If $f(x) = |x|$, then the graph of $y = -f(x + 2)$ looks like:
(a) True (b) False



Exercise 9

10. If $f(x) = |x|$, then the graph of $y = |f(x) - 2|$ looks like:
(a) True (b) False



Exercise 10

Solutions

1(b) 2(d) 3(d) 4(c) 5(a) 6(b) 7(b) 8(b) 9(a) 10(a)