LESSON 19: MEAN VALUE THEOREM

Keywords
Mean Value Theorem

Let's look at the situation where: You are driving on a highway, and there are two police stations on the way. Of course, like a normal person, you slow down to the speed limit, say 55 miles/hour. Then, as soon as you are out of sight of the station, you speed up and slow down again only when you are near the second police station. A traffic cop stops your car and asks you the following questions, to which you reply truthfully:

"When did you go through the first police station?"
"About an hour ago."
"Did you get a ticket?"
"No. I was going at 55 miles/hour."
"The two police stations are 80 miles apart from each other. You get a ticket."
"But I was driving at 55 miles/hour when you stopped me!"

What happens here is that the cop computed your average velocity between the two stations as 80 miles/hour. Even though you were driving at 55 miles/hour near the police stations, it still means that in between these stations there was a point where you were driving with an instantaneous velocity of 80 miles/hour, and therefore you deserve a ticket.

The above situation is not to show that one should lie to a traffic cop, but to show an application of the *Mean Value Theorem*, which basically states that if your average velocity during a certain interval of time was x miles/hour, then you must have had at least one point in time where your instantaneous velocity was x miles/hour.

Here is the formal statement of the *Mean Value Theorem*.

If f is continuous on a closed interval $[a, b]$ and differentiable on the interval (a, b), then there is at least one number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Notice that if f was the position function, then $\frac{f(b)-f(a)}{b-a}$ would be the formula for the average velocity between times a and b and $f'(c)$ would indicate the instantaneous velocity at time c.

Examples

1. If f is continuous on $[a, b]$ and differentiable on (a, b) and if $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.

   Since $f(a) = f(b)$ and f satisfies the conditions of the Mean Value Theorem, we have $\frac{f(b)-f(a)}{b-a} = 0$ and the Mean Value Theorem guarantees that there is at least one point c in (a, b) where $f'(c) = \frac{f(b)-f(a)}{b-a} = 0$.

2. Find the number c guaranteed by the Mean Value Theorem for $f(x) = 2\sqrt{x}$ on $[1, 4]$.

   We first check that f is continuous on $[1, 4]$ and differentiable on $(1, 4)$. The Mean Value Theorem says that there is a c in $(1, 4)$ such that $f'(c) = \frac{f(4)-f(1)}{4-1}$.

   Since $f'(x) = \frac{1}{\sqrt{x}}$, we need to find a c such that $f'(c) = \frac{1}{\sqrt{c}} = \frac{2\sqrt{4}-2\sqrt{1}}{4-1} = \frac{2}{3}$. It is not hard to see that $c = \frac{9}{4}$.

3. Another application of the Mean Value Theorem is the proof of our criteria to find the monotonicity of f in the previous lesson on *Monotonicity and Concavity*.

   We suppose that f is continuous on an interval I and that $f'(x) > 0$ at each point x in the interior of I. We want to show that f is increasing on I.

Consider any two points x_1 and x_2 of I, with $x_1 < x_2$. By the Mean Value Theorem applied to the interval $[x_1, x_2]$, there is a point c in (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

or, equivalently, such that

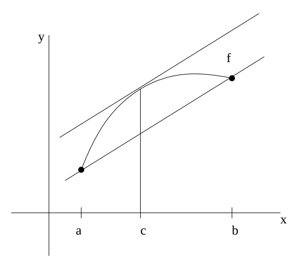
$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

Since $f'(c) > 0$ by assumption, and $x_2 > x_1$, we see that $f(x_2) - f(x_1) > 0$, that is, $f(x_2) > f(x_1)$. This is the same as saying that f is increasing on I.

In a similar way we show that if $f'(x) < 0$ for all x in the interior of I, then f is decreasing on I.

Graphical Interpretation of The Mean Value Theorem

We notice first that $\frac{f(b)-f(a)}{b-a}$ is the slope of the line connecting the points $(a, f(a))$ and $(b, f(b))$ on the graph. The Mean Value Theorem therefore states that between a and b there is at least one point c such that the slope of the tangent line to the graph of f at this point is equal to $\frac{f(b)-f(a)}{b-a}$, the slope of the line connecting $(a, f(a))$ and $(b, f(b))$.



Exercises

For each of the exercises 1-5, a function is defined and a close interval is given. Decide whether the Mean Value Theorem applies to the given function on the given interval. If it does, find all possible values of c that satisfy the theorem.

1. $f(x) = |x|$; $[1, 2]$.
   (a) does not satisfy (b) $c = 1$ (c) $c = 2$ (d) all points in $(1, 2)$
2. $g(x) = |x|$; $[-2, 2]$.
   (a) does not satisfy (b) $c = 0$ (c) $c = -2$ (d) $c = 1$
3. $h(x) = \frac{x}{x-3}$; $[0, 2]$.
   (a) does not satisfy (b) $c = 3 - \sqrt{3}$ (c) $c = 12$ (d) $c = 3 - \sqrt{3}, 3 + \sqrt{3}$
4. $k(x) = \frac{x}{x-3}$; $[0, 4]$.
   (a) does not satisfy (b) $c = 3 - \sqrt{3}$ (c) $c = 12$ (d) $c = 1$
5. $F(t) = t^2 + 3t - 1$; $[-3, 1]$.
   (a) does not satisfy (b) $c = \frac{1}{2}$ (c) $c = -\frac{5}{2}$ (d) $c = -\frac{5}{4}$

Solutions

1(d) 2(a) 3(b) 4(a) 5(d)