LESSON 18: SOPHISTICATED GRAPHING

Keywords

check: domain, range, x-intercept, y-intercept, limits at infinity, monotonicity, concavity, inflection points, local maximum, local minimum

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Now that we have seen that the first and second derivatives of a function can give information about where the function is increasing, decreasing, concave up, convave down, and where it attains its maximum and minimum values (if any), we can add up all the techniques and obtain valuable information that will allow us to have a very good idea of how the graph of the given function should look like.

A polynomial function of degree 1 or 2 is easy to graph by hand; and of course no one should be asked to graph a polynomial of degree 15 or higher! But if the degree of the polynomial function is between 3 and 6, one can try plotting its graph using calculus.

Let's sketch the graph of $f(x) = 2x^3 - 3x - 10$. First we try to figure out where $f(x) = 0$ to find the x-intercept. When this is not possible, one can compute $f(0)$ in order to have a reference as to where the graph of f intercept the y-axis. In this example $f(0) = -10$.

We also check that the domain of f is the whole real line, that is, there are no numbers that f cannot accept as possible inputs. Notice that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$. Since f is continuous everywhere, this implies that the range of f is also the whole set of real numbers.

Now we take the first derivative of f and get $f'(x) = 6x^2 - 3$. Hence $f'(x) = 0$ when $x = \pm\frac{\sqrt{2}}{2}$, and $f'(x) > 0$ on $\left(-\infty, -\frac{\sqrt{2}}{2}\right) \cup \left(\frac{\sqrt{2}}{2}, \infty\right)$ and $f'(x) < 0$ on $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Therefore, f is increasing on $\left(-\infty, -\frac{\sqrt{2}}{2}\right) \cup \left(\frac{\sqrt{2}}{2}, \infty\right)$ and decreasing on $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

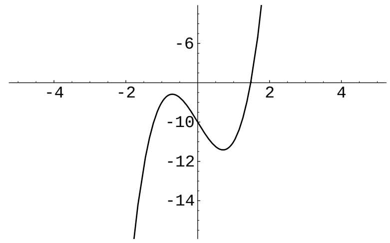
If we take the second derivative of f we get $f''(x) = 12x$. Therefore, f is concave up if $x > 0$ and concave down if $x < 0$, and $x = 0$ is an inflection point.

It is always convenient to compute the values of f at the points where f' and f'' are equal to zero. In this case, the points are $-\frac{\sqrt{2}}{2}$, 0, and $\frac{\sqrt{2}}{2}$.

Adding up all the information, we now have:

- f is continuous and its domain is the set of all real numbers;
- $f(0) = -10$;
- $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$;
- f is increasing and concave down on $\left(-\infty, -\frac{\sqrt{2}}{2}\right)$;
- f is increasing and concave up on $\left(\frac{\sqrt{2}}{2}, \infty\right)$;
- f is decreasing and concave down on $\left(-\frac{\sqrt{2}}{2}, 0\right)$;
- f is decreasing and concave down on $\left(0, \frac{\sqrt{2}}{2}\right)$;
- $f\left(-\frac{\sqrt{2}}{2}\right) = -10 + \sqrt{2}$ and $f\left(\frac{\sqrt{2}}{2}\right) = -10 - \sqrt{2}$.

The graph of f therefore looks like



$f(x) = 2x^3 - 3x - 10$

We can tell from the graph that f has only one root, that is, there is only one point c such that $f(c) = 0$, and $c > \frac{\sqrt{2}}{2}$.

Notice that the function f is increasing before $-\frac{\sqrt{2}}{2}$ and it is decreasing afterwards. That is, there is a change of monotonicity of f from increasing to decreasing at the point $x = -\frac{\sqrt{2}}{2}$. All points that lie near the point $x = -\frac{\sqrt{2}}{2}$ have values lower than $f\left(-\frac{\sqrt{2}}{2}\right) = -10 + \sqrt{2}$. Therefore, $\left(-\frac{\sqrt{2}}{2}, -10 + \sqrt{2}\right)$ is called a *local maximum point.*

A point c is called a *local maximum point* and its corresponding $f(c)$ is called a *local maximum value* of f if there is an interval (a, b) containing c such that $f(c)$ is the maximum value of f on the interval (a, b).

There is a difference between a local maximum value and the maximum value that we studied in Lesson 11. There the maximum value was *global*, that is, $f(c)$ was a maximum value if $f(c) \geq f(x)$ for all x in the domain of f. Here we are dealing only with a local property; $f(c)$ is only greater than the values of its neighbours, not of all other points in the domain.

A method of checking whether a differentiable function f has a local maximum point at c is to see whether the sign of f' changed from positive to negative at point $x = c$. That would mean f was increasing before $x = c$, and decreasing afterwards, hence $x = c$ would be a local maximum point.

We can define local minimum point and local minimum value in a similar fashion.

A point c is called a *local minimum point* and its corresponding $f(c)$ is called a *local minimum value* of f if there is an interval (a, b) containing c such that $f(c)$ is the minimum value of f on the interval (a, b).

To check whether a differentiable function f has a local minimum point at c, we can see whether the sign of f' changed from negative to positive at c. That would mean f is decreasing before $x = c$ and increasing after $x = c$, hence $x = c$ would be a local minimum point.

Note: Using the first derivative to check whether a function has a local maximum or local minimum at a point only works, of course, if the function is differentiable. Also, since we are looking for changes on the sign of f', the best candidates for local maximum/minimum points are the points where $f' = 0$.

In the example $f(x) = 2x^3 - 3x - 10$, f has a local minimum at $\frac{\sqrt{2}}{2}$.

We mention here that a global maximum is always a local maximum, and a global minimum is always a local minimum, but the converse might not be true.

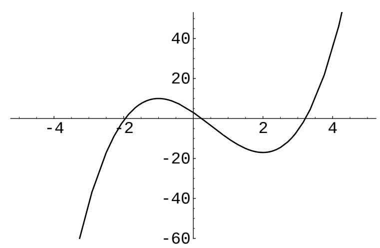
Exercises

1. Sketch the graph of $f(x) = 2x^3 - 3x^2 - 12x + 3$.

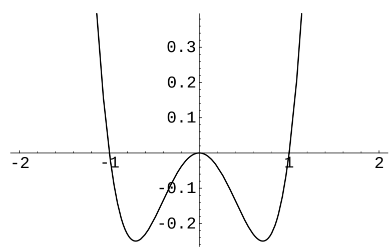
2. Sketch the graph of $f(x) = x^2(x^2 - 1)$.

3. What are the local minimum points of $g(x) = x^4 + x^2 - 3$?
   (a) $x = -3$ (b) $x = 1$ (c) $x = 0, -3$ (d) $x = 0$
4. What are the local maximum points of $h(t) = t + \frac{1}{t}$, $t \neq 0$?
   (a) $t = -1, 1$ (b) $t = -1$ (c) $t = 1$ (d) no local maximum points
5. What are the local minimum points of $h(t) = t + \frac{1}{t}$, $t \neq 0$?
   (a) $t = -1, 1$ (b) $t = -1$ (c) $t = 1$ (d) no local minimum points

Solutions



Exercise 1



Exercise 2

3(d) 4(b) 5(c)