LESSON 17: MONOTONICITY AND CONCAVITY

Keywords
increasing, decreasing, non-decreasing, non-increasing, monotonic, strictly monotonic, concave up, concave down, inflection point

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The previous lesson and this one will help us in the next one, on *Sophisticated Graphing* of functions.

Here we define what it means for f to be increasing/decreasing/monotonic, and how does the graph of a concave up/down function looks like.

Monotonicity

Let f be defined on an interval (open, closed, or neither). We say that

(i) f is *increasing* on the interval if, for every pair of numbers x_1 and x_2 such that $x_1 < x_2$, we have $f(x_1) < f(x_2)$;

(ii) f is *decreasing* on the interval if, for every pair of numbers x_1 and x_2 such that $x_1 < x_2$, we have $f(x_1) > f(x_2)$;

(iii) f is *non-decreasing* on the interval if, for every pair of numbers x_1 and x_2 such that $x_1 < x_2$, we have $f(x_1) \leq f(x_2)$;

(iv) f is *non-increasing* on the interval if, for every pair of numbers x_1 and x_2 such that $x_1 < x_2$, we have $f(x_1) \geq f(x_2)$;

(v) f is *monotonic* if f is either non-decreasing or non-increasing on the interval (the "or" is exclusive);

(vi) f is *strictly monotonic* if f is either increasing or decreasing on the interval (the "or" being exclusive here).

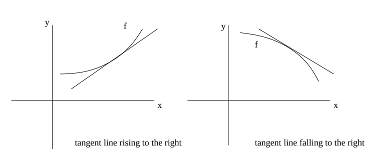
There is a way to tell whether a function is increasing or decreasing.

Let f be a continuous function defined on an interval I and differentiable at every interior point of I (this is just in case I is closed at any of its endpoints).

(i) If $f'(x) > 0$ for all x an interior point of the interval I, then f is increasing on I.

(ii) If $f'(x) < 0$ for all x an interior point of the interval I, then f is decreasing on I.

The above (which we shall not prove) can be understood intuitively if we remember that the first derivative of f at x gives the value of the slope of the tangent line of the graph of f at x. If $f'(x) > 0$, the tangent line is rising to the right, so we kind of expect the graph of f to be rising to the right (increasing) also. If $f'(x) < 0$, the tangent line is falling to the right, and therefore we expect the graph of f to be falling (decreasing) too.



Examples:

1. If $f(x) = 2x^3 - 9x^2 + 12x$, find where (which intervals) f is increasing and where it is decreasing.

   We begin by finding the derivative of f.

   $$f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2).$$

   Then we see where f' is positive and where it is negative. $f'(x) = 6(x-1)(x-2)$ is a parabola "facing up" (like a cup) with roots at 1 and 2. Therefore, f' is positive for numbers smaller than 1 and for numbers greater than 2, but it is negative between 1 and 2. Our solution is:

   f is increasing on $(-\infty, 1) \cup (2, \infty)$ and f is decreasing on $(1, 2)$.

2. If $g(x) = \frac{x-1}{x^2}$, where is g increasing and where it is decreasing?

   Using the quotient rule for derivatives, we have

   $$g'(x) = \frac{x^2 \cdot 1 - 2x \cdot (x-1)}{x^4} = \frac{-x^2 + 2x}{x^4}.$$

   Now we check where $g'(x) > 0$ and where $g'(x) < 0$. Notice that $x^4 > 0$ for all $x \neq 0$ (and 0 is not in the domain of g anyway), so the sign of the numerator defines the sign of g'. The numerator $2x - x^2 = x(2-x)$ is another parabola, this time "cup down", with roots at 0 and 2. Therefore $x(2-x)$ is positive between 0 and 2, and negative if $x < 0$ or $x > 2$. Hence $g'(x) > 0$ if x is in $(0, 2)$, and $g'(x) < 0$ if $x < 0$ or $x > 2$.

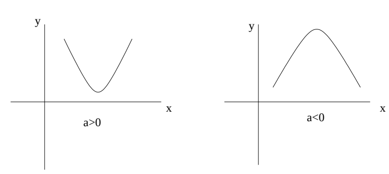
   We conclude that g is increasing on $(0, 2)$ and decreasing on $(-\infty, 0) \cup (2, \infty)$.

Concavity

Look at a random parabolic function, that is, a function of the form

$$f(x) = ax^2 + bx + c,$$

where $a \neq 0$.



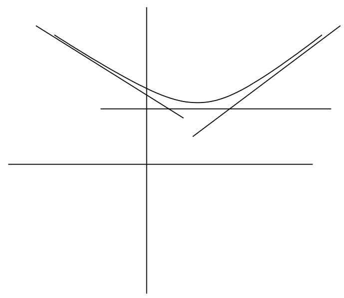
$$f(x) = ax^2 + bx + c$$

We know that the graph of f will be "cup up" if $a > 0$, and "cup down" if $a < 0$. It is not a coincidence that the second derivative of f, that is, f'' (the second derivative of f is the derivative of the derivative of f), is equal to a. We will soon see that this property of the graph being "cup up" or "cup down" can be determined by the sign of f''.

We know give two new definitions. Let f be differentiable on an open interval I. We say that f (as well as its graph) is *concave up* on I if f' is increasing on I, and we say that f is *concave down* on I if f' is decreasing on I.

Notice that the fact that f' is increasing DOES NOT imply f is increasing, since f' could very well be increasing and negative on an interval, which would imply on f being decreasing instead.

The picture below shows how a graph of f with increasing f' will take a shape that looks like a cup (f' increasing means that the slopes of the tangent lines are increasing. Notice that the "left side" of the cup has tangent lines with negative slopes that are approaching slope zero, and the



"right side" of the cup has tangent lines with positive and increasing slopes. Similarly, the graph of a function f with decreasing f' will look like an inverted cup (concave down).

We just saw that, if f is a differentiable function, then if f' is always positive on an interval then f is increasing on the same interval. Therefore, if f' is a differentiable function, then if f'' is always positive on an interval then f' is increasing on the same interval. From this we get the following:

Let f be twice differentiable (that is, both f' and f'' exist) on the open interval I.

(i) If $f''(x) > 0$ for all x in the interval I, then f is concave up on I.

(ii) If $f''(x) < 0$ for all x in the interval I, then f is concave down on I.

Examples:

1. Where is $f(x) = (x-1)^2$ increasing, decreasing, concave up and concave down?

   To find where f is increasing/decreasing, we need to find f'.

   $$f'(x) = 2(x-1),$$

   which is negative for $x < 1$ and positive for $x > 1$. Therefore, f is decreasing on $(-\infty, 1)$ and is increasing on $(1, \infty)$.

   To find where f is concave up/down, we need to find f'', which is $f''(x) = 2$. Since $2 > 0$, this means f is always concave up.

   Notice that $f(x) = (x-1)^2 = x^2 - 2x + 1$, so the coefficient that accompanies the term x^2 is 1, which is positive. As we mentioned before, a parabolic function will be concave up if it is of the form $ax^2 + bx + c$ with $a > 0$, and it will be concave down if $a < 0$.

2. Where is $g(x) = \frac{x}{1+x^2}$ concave up and where it is concave down?

   We have $g'(x) = \frac{(1+x^2)\cdot 1 - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$ and

   $$\begin{aligned} g''(x) &= \frac{(1+x^2)(-2x) - (1-x^2)(2)(1+x^2)(2x)}{(1+x^2)^4} \\ &= \frac{(1+x^2)[(1+x^2)(-2x) - (1-x^2)(4x)]}{(1+x^2)^4} \\ &= \frac{2x^3 - 6x}{(1+x^2)^3} = \frac{2x(x^2-3)}{(1+x^2)^3} \end{aligned}$$

   We notice here that the denominator of g'', $(1+x^2)^3$, is always positive. To determine the sign of g'' we need to determine only the sign of the numerator, $2x(x^2-3)$. The roots are $-\sqrt{3}$, 0 and $\sqrt{3}$. If $x < \sqrt{3}$, we see that $2x(x^2-3)$ is negative. If $-\sqrt{3} < x < 0$, then $2x$ is negative, but $x^2 - 3$ is also negative, hence $2x(x^2-3)$ is positive. If $0 < x < \sqrt{3}$, then $2x(x^2-3)$ is negative, and $2x(x^2-3)$ is positive when $x > \sqrt{3}$. So we have

- g is concave up on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$;
- g is concave up on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$.

The graph of g is shown in Figure 1.

Inflection Point

Here is our last definition for this lesson: Let f be continuous at c. We call $(c, f(c))$ an *inflection point* of the graph of f if f changes concavity at c, that is, f is concave up on one side of c and concave down on the other side.

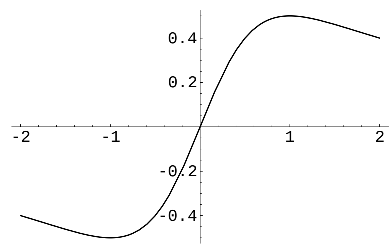


Figure 1

Notice that f does not need to be differentiable at c. Points where $f''(x) = 0$ or where $f''(x)$ does not exist are the *candidates* for points of inflection (we need to check, though).

Examples

1. The point $x = 0$ is an inflection point for $f(x) = x^3$.

   We have $f''(x) = 6x$, which is positive for $x > 0$ and negative for $x < 0$. Hence the function is concave down before zero and concave up after zero. Zero is an inflection point.

2. The function $f(x) = \sin x$ has inflections points at $k\pi$, where k is an integer.

   Notice that $f''(x) = -\sin x$, which is equal to zero at $k\pi$, where k is an integer. Let's look at the function near $x = \pi$. For numbers x near π but smaller than π, we have $-\sin x < 0$. For numbers x near π but greater than π, we have $-\sin x > 0$. Therefore, $(\pi, \sin\pi)$ is an inflection point of f. One can show in a similar fashion that all integer multiples of π are inflection points of f.

Exercises

1. If $h(z) = \frac{z^4}{4} - \frac{4z^3}{6}$, where is h increasing?

   (a) $(2, \infty)$ (b) $(-\infty, 2)$, (c) $(0, 2)$ (d) $(0, \infty)$

2. If $f(x) = x^2 - \frac{1}{x^2}$, where is f decreasing?

   (a) nowhere (b) $(-\infty, 0)$ (c) $(0, \infty)$ (d) $(-\infty, 0) \cup (0, \infty)$

3. If $g(x) = 3x^5 - 5x^3 + 1$, where is the graph of g concave down?

   (a) nowhere (b) $(-\infty, 0)$ (c) $(-\frac{\sqrt{2}}{2}, 0) \cup (\frac{\sqrt{2}}{2}, \infty)$ (d) $(-\infty, -\frac{\sqrt{2}}{2}) \cup (0, \frac{\sqrt{2}}{2})$

4. If $g(x) = 3x^5 - 5x^3 + 1$ (same as above), what are the inflection points?

   (a) 0 (b) $-\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{2}}{2}$ (c) $-\frac{\sqrt{2}}{2}$, 0 and $\frac{\sqrt{2}}{2}$ (d) none

5. A function cannot be concave up and decreasing on the same interval.
   (a) True (b) False
6. Suppose that $f'' > 0$ and $g'' > 0$ for all x. Then $f(x) + g(x)$ is concave up for all x.
   (a) True (b) False
7. Suppose that $f'' > 0$ and $g'' > 0$ for all x. Then $f(x) \cdot g(x)$ is concave up for all x.
   (a) True (b) False
8. Find all inflection points of $T(t) = 3t^3 - 18t$.
   (a) none (b) 0 (c) $\sqrt{2}$ (d) $-\sqrt{2}$ and $\sqrt{2}$

Solutions

1(a) 2(b) 3(d) 4(c) 5(b) 6(a) 7(b) 8(b)