LESSON 16: MAXIMUM AND MINIMUM PROBLEMS

Keywords
maximum value, maximum point, minimum value, minimum point, extreme value, extreme point, endpoint of interval, stationary point, singular point, critical point

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In this lesson we start our first real application of derivatives (if we do not count the fact that the derivative of the position function gives us the instantaneous velocity of a car). Our motivation will be the following problem:

We have 200 feet of fence with which we plan to enclose a rectangular pig pen in our farm. If we wish to enclose the maximum area (so that the pigs will be as comfortable as possible), how should we proceed?

The solution to this problem would be to enclose a square pig pen, which would give us $\left(\frac{200}{4}\right)^2 =$ 2500 square feet, the maximum area (under the constraint that the pen needs to be rectangular in shape). If there were no restrictions on the shape of the pen, a circular one would give the maximum area, $\frac{100^2}{\pi} \geq 3000$ square feet. However, to solve the problem without restricting the shape of the pig pen is much harder and beyond the scope of this study guide).

First, let's recall what it means for a number to be a maximum/minimum point of a function (we already mentioned this in the lesson *Some Properties of Continuous Functions or The Intermediate Value Theorem*).

Let f be a function and c a point in its domain. We say that $f(c)$ is the *maximum value* of f if $f(c) \geq f(x)$ for every x in the domain of f. In this case, c is called a *maximum point* of f.

Analogously, we say that $f(c)$ is the *minimum value* of f if $f(c) \leq f(x)$ for all x in the domain of f, and c is called a *minimum point* of f.

Moreover, $f(c)$ is called an *extreme value* of f if it is either the maximum value or the minimum value (the corresponding c being called an *extreme point*).

Recall that on a closed interval a continuous function f always have a maximum and a minimum value (they could be the same if f is a constant function).

Where do the extreme values of f occur?

Let f be defined on an interval containing the point c. If $f(c)$ is an extreme value, then c must be either

(i) an endpoint of the interval; or

(ii) a point where $f'(c) = 0$; or

(iii) a point where $f'(c)$ does not exist.

The proof of this goes as follows.

Let's assume $f(c)$ is the maximum value of f on the interval and suppose c is not an endpoint nor a point where the derivative of f does not exist. We will show that in this case, c must be such that $f'(c) = 0$. That would prove our previous statement.

Now, since $f(c)$ is the maximum value, $f(x) \leq f(c)$ for all x in the interval, hence $f(x) - f(c) \leq 0$ always. If $x < c$, then

$$\frac{f(x) - f(c)}{x - c} \geq 0,$$

since $x - c < 0$ if $x < c$.

On the other hand, if $x > c$, then $x - c > 0$ and

$$\frac{f(x) - f(c)}{x - c} \leq 0.$$

We know (by assumption) that $f'(c)$ exist. Then the two limits $\lim_{x\to c^-} \frac{f(x)-f(c)}{x-c}$ and $\lim_{x\to c^+} \frac{f(x)-f(c)}{x-c}$ must exist and be equal.

Since $\frac{f(x)-f(c)}{x-c} \geq 0$ for $x < c$, the limit $\lim_{x\to c^-} \frac{f(x)-f(c)}{x-c}$ must also be ≥ 0. Similarly, we have $\lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} \leq 0$.

As $\lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c^+} \frac{f(x)-f(c)}{x-c}$, they must be equal to zero, and $f'(c) = 0$.

The proof is analogous if we had assumed in the beginning that $f(c)$ is the minimum value of f in the interval.

Now, to find the extreme values of a function f on an interval, we need only check the endpoints of the interval, the points where f' is equal to zero, and the points where f' does not exist.

A point c such that $f'(c) = 0$ is called a *stationary point*; a point c such that $f'(c)$ does not exist is called a *singular point*, and a point c is called a *critical point* if it is either an endpoint of the interval where f is defined, or a stationary point, or a singular point. To find the extreme values of f we proceed as follows: we search for the critical points of f, and then evaluate f at these points, choosing the highest value as the maximum value and the lowest, as the minimum value.

Examples:

1. Find the maximum and minimum values of $f(x) = -2x^3 + 3x^2$ on the interval $[-1, \frac{3}{2}]$.

   Since $f'(x) = -6x^2 + 6x = -6x(x-1)$, we have $f'(0) = 0$ and $f'(1) = 0$. Therefore the critical points of f are:

   - endpoints: -1 and $\frac{3}{2}$
   - stationary points: 0 and 1
   - singular points: no singular points, since f is differentiable everywhere (a polynomial function is always differentiable everywhere).

   Now we compute $f(-1)$, $f(\frac{3}{2})$, $f(0)$, $f(1)$ and compare. We have: $f(-1) = 5$, $f(\frac{3}{2}) = 0$, $f(0) = 0$, and $f(1) = 1$. Hence the minimum value is 0 and the maximum value is 5.

2. Let's go back to our motivational problem. We have 200 feet of fence with wich we plan to enclose the largest area rectangular shaped pig pen possible. How should we do this?

   Apparently, these types of problems (the so called word problems) scare people, because we do not have a function f to work with; we need to find our own f. It is not as hard as it may seem. We work with all the information we have. The only things we know are:

   - pig pen should be rectangular in shape
   - the fence should be 200 feet long, that is, the perimeter of the rectangular pig pen should be 200 feet.

   Call the length of the pig pen x and the width of the pig pen y. We know that $2x + 2y = 200$ feet. We want the maximum area possible for the pen, that is, we want xy to be a maximum. So we want to find the maximum value for $f(x) = xy$. This "y" inside the expression of f is not very nice. We do not want to have two variables inside our function. But y is not really a variable. For example, if we say that the pen has length 10 feet, we immediately conclude that the width y of the pig pen is 90 feet, because the total perimeter of the pen should be 200 feet. In other words, y is determined by the relation $2x + 2y = 200$, once x is given. That is, $y = \frac{200-2x}{2} = 100 - x$.

   Then our function f becomes $f(x) = x(100 - x)$. To find the maximum value of this function, we should look at its critical points and then evaluate f at these points. Notice that we can consider only the values of x that lie between 0 and 100. For $x < 0$ or $x > 100$, we have $f(x)$ negative, which would mean negative area for the pig pen and that does not relate to our problem (we want the pigs to have positive area to run around; what would negative area mean anyway?). Hence the interval we are considering can be taken as $[0, 100]$. The critical points of f are:

- endpoints: 0 and 100
- stationary points: where f' equals zero. We have $f'(x) = (100 - x) + x(-1) = 100 - 2x$. The stationary point is $x = 50$.
- critical points: there are none.

Evaluating f at 0, 100 and 50 and comparing, we get $f(0) = 0$, $f(100) = 0$ and $f(50) = 2500$. Therefore, 2500 square feet is the maximum value for our function, that is, 2500 feet is the maximum area our rectangular pig pen can have. Notice that if the length of our pig pen is 50 feet, that means the pen is square.

Exercises

1. Given $f(x) = x^2 + 4x + 4$, on the interval $[-1, 4]$, what are the critical points of f?

   (a) no critical points (b) -1 and 4 (c) -2 (d) $-1, -2$ and 4

2. What is the maximum and minimum values of $f(x) = x^2 + 4x + 4$ on $[-1, 4]$?

   (a) 4 and 36 (b) 0 and 36 (c) 1 and 36 (d) -1 and 4

3. A farmer has 100 feet of fence with which he plans to enclose a rectangular yard for his alligator. If he wishes to enclose maximum area, what should the dimensions be?

   (a) 15×35 (b) 20×30 (c) 25×25 (d) none of the answers

4. What positive number exceeds its cube by the maximum amount? (Hint: notice that $x > x^3 > 0$ only if $0 < x < 1$, hence one can consider f on the interval $[0, 1]$)

   (a) 1 (b) $\frac{1}{2}$ (c) $\frac{\sqrt{3}}{3}$ (d) none of the answers

5. What are the nonnegative numbers whose sum is 10 and whose product is a maximum?

   (a) 5 and 5 (b) 4 and 6 (c) $5\sqrt{2}$ and $10 - 5\sqrt{2}$ (d) none of the answers

6. What are the critical points of the function $f(t) = \sin t - \cos t$ on the interval $[0, 2\pi]$?

   (a) $0, 2\pi$ (b) $0, \pi, 2\pi$ (c) $0, \frac{3\pi}{4}, \frac{7\pi}{4}, 2\pi$ (d) $0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}, 2\pi$

7. What is the minimum value of $f(t) = \sin t - \cos t$ on $[0, \pi]$?

   (a) 0 (b) -1 (c) $-\frac{\sqrt{2}}{2}$ (d) 1

8. What is the minimum value of $f(t) = \sin t - \cos t$ on $[0, 2\pi]$?

   (a) $\sqrt{2}$ (b) 1 (c) 0 (d) $\frac{\sqrt{3}}{2}$

Solutions

1(d) 2(b) 3(c) 4(c) 5(a) 6(c) 7(b) 8(a)