LESSON 15: L'HOPITAL'S RULE

Keywords

Limits of the form $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, and $\infty - \infty$

We shall not prove this rule; but since it is a very useful tool for finding many indeterminate limits, one should learn how to apply the rule well.

L'Hopital's rule is a rule for finding many (but not all!) indeterminate limits of the form $\frac{0}{0}$, $\frac{\infty}{\infty}$ or $\frac{-\infty}{\infty}$. For example, for limits like

$$\lim_{x\to 0} \frac{\sin x}{x}, \quad \lim_{x\to 1} \frac{x^2-1}{x-1}, \quad \lim_{x\to\infty} \frac{2x^5-x^4+3}{3x^3+5},$$

when we plug in the respective values of x at the limit point, we get $\frac{0}{0}$, $\frac{0}{0}$ and $\frac{\infty}{\infty}$, respectively. We have seen how to find the values of these limits before. We know that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1} x+1 = 2$, and $\lim_{x\to\infty} \frac{2x^5-x^4+3}{3x^3+5} = \lim_{x\to\infty} \frac{2x^2-x+\frac{3}{x^3}}{3+\frac{5}{x^3}} = \infty$.

L'Hopital's rule, however, will allow us to find these limits without the use of "tricks" like factoring or dividing numerator and denominator by x^3.

L'Hopital's rule

Suppose that $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ (or $|\lim_{x\to c} f(x)| = |\lim_{x\to c} g(x)| = \infty$).

If $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists in either the finite or infinite sense (that is, if the limit is a finite number or ∞ or $-\infty$), then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

We have similar statements of L'Hopital's rule for left and right hand limits ($x \to c^-$ or $x \to c^+$) and also for $x \to -\infty$ or $x \to \infty$.

Remember: L'Hopital's rule ONLY applies to limits of the form $\frac{0}{0}$, $\frac{\infty}{\infty}$ or $\frac{-\infty}{\infty}$.

Examples:

1. We can use L'Hopital's rule to show that $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$.

   When we plug in $x = 0$ in the fraction $\frac{1-\cos x}{x}$, we get $\frac{0}{0}$, hence we can apply L'Hopital's rule to this limit:

   $$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{(1-\cos x)'}{(x)'} = \lim_{x\to 0} \frac{0-+\sin x}{1} = \sin 0 = 0$$

   Therefore, if we remember L'Hopital's rule, we do not need to memorize the limits $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$ anymore!

2. Find $\lim_{x\to 1} \frac{x^2-1}{x-1}$ using L'Hopital's rule.

   Since this limit is of the form $\frac{0}{0}$, it is OK to use L'Hopital's rule here.

   $$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x^2-1)'}{(x-1)'} = \lim_{x\to 1} \frac{2x}{1} = 2.$$

3. Find $\lim\limits_{h\to\infty} \frac{2h^5-h^4+3}{3h^3+5}$.

This limit is of the form $\frac{\infty}{\infty}$, therefore we can use L'Hopital's rule.

$$\lim_{h\to\infty} \frac{2h^5-h^4+3}{3h^3+5} = \lim_{h\to\infty} \frac{(2h^5-h^4+3)'}{(3h^3+5)'} = \lim_{h\to\infty} \frac{10h^4-4h^3}{9h^2} = \lim_{h\to\infty} \frac{10h^2-4h}{9} = \infty.$$

4. Find $\lim\limits_{\theta\to 0} \frac{\sin\theta-\theta}{\theta^3}$.

This limit is of the form $\frac{0}{0}$, therefore we can use L'Hopital's rule and get

$$\lim_{\theta\to 0} \frac{\sin\theta-\theta}{\theta^3} = \lim_{\theta\to 0} \frac{(\sin\theta-\theta)'}{(\theta^3)'} = \lim_{\theta\to 0} \frac{\cos\theta-1}{3\theta^2}.$$

But $\lim\limits_{\theta\to 0} \frac{\cos\theta-1}{3\theta^2}$ is again a limit of the form $\frac{0}{0}$! What do we do now? We use L'Hopital's rule again!

$$\lim_{\theta\to 0} \frac{\cos\theta-1}{3\theta^2} = \lim_{\theta\to 0} \frac{(\cos\theta-1)'}{(3\theta^2)'} = \lim_{\theta\to 0} \frac{-\sin\theta}{6\theta}.$$

Again, the limit $\lim\limits_{\theta\to 0} \frac{-\sin\theta}{6\theta}$ is of the form $\frac{0}{0}$. We can apply L'Hopital's rule for the third time and arrive at

$$\lim_{\theta\to 0} \frac{-\sin\theta}{6\theta} = \lim_{\theta\to 0} \frac{(-\sin\theta)'}{(6\theta)'} = \lim_{\theta\to 0} \frac{-\cos\theta}{6} = \frac{-1}{6}.$$

Therefore, $\lim\limits_{\theta\to 0} \frac{\sin\theta-\theta}{\theta^3} = -\frac{1}{6}$.

In the last step, we could also have noticed that $\lim\limits_{\theta\to 0} \frac{-\sin\theta}{6\theta} = -\frac{1}{6} \lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = -\frac{1}{6}$, since $\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

5. Find $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2+3x}$.

Since this is a limit of the form $\frac{0}{0}$, we can use L'Hopital's rule.

$$\lim_{x\to 0} \frac{1-\cos x}{x^2+3x} = \lim_{x\to 0} \frac{(1-\cos x)'}{(x^2+3x)'} = \lim_{x\to 0} \frac{\sin x}{2x+3}.$$

Here we STOP using L'Hopital's rule, because the limit $\lim\limits_{x\to 0} \frac{\sin x}{2x+3}$ is NOT of the form $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$.

$$\lim_{x\to 0} \frac{\sin x}{2x+3} = 0.$$

(when we plug in $x = 0$ into the expression, we get 0 as the numerator and $2\cdot 0+3 = 3$ as the denominator.)

L'Hopital's rule does not solve ALL indeterminates limits (otherwise that would make our lives too easy)! Here is an example where L'Hopital's rule does not help.

Find $\lim\limits_{x\to\frac{\pi}{2}} \frac{\tan x}{\sec x}$.

Recall that $\tan\frac{\pi}{2} = \sec\frac{\pi}{2} = \infty$, hence we could try using L'Hopital's rule to get

$$\lim_{x\to\frac{\pi}{2}} \frac{\tan x}{\sec x} = \lim_{x\to\frac{\pi}{2}} \frac{(\tan x)'}{(\sec x)'} = \lim_{x\to\frac{\pi}{2}} \frac{\sec^2 x}{\sec x\tan x} = \lim_{x\to\frac{\pi}{2}} \frac{\sec x}{\tan x}.$$

Again we have a limit of the form $\frac{\infty}{\infty}$. We could apply L'Hopital's rule to the limit $\lim\limits_{x\to\frac{\pi}{2}} \frac{\sec x}{\tan x}$ and get

$$\lim_{x\to\frac{\pi}{2}} \frac{\sec x}{\tan x} = \lim_{x\to\frac{\pi}{2}} \frac{(\sec x)'}{(\tan x)'} = \lim_{x\to\frac{\pi}{2}} \frac{\sec x\tan x}{\sec^2 x} = \lim_{x\to\frac{\pi}{2}} \frac{\tan x}{\sec x}.$$

This brings us back to the same expression we started off with, which means that L'Hopital's rule is not going to take us anywhere in this case. The way to find $\lim_{x\to\frac{\pi}{2}} \frac{\tan x}{\sec x}$ is

$$\lim_{x\to\frac{\pi}{2}} \frac{\tan x}{\sec x} = \lim_{x\to\frac{\pi}{2}} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} = \lim_{x\to\frac{\pi}{2}} \frac{\sin x}{\cos x} \cdot \cos x = \lim_{x\to\frac{\pi}{2}} \sin x = 1.$$

Other Indeterminate Forms

What about other indeterminate forms, for example, of the type $\infty - \infty$ or of the type $0 \cdot \infty$? How do we compute $\lim_{x\to 0} 3x^2 \csc^2 x$ (this is a limit of the form $0 \cdot \infty$)?

In these cases, we can still use L'Hopital's rule but only after a proper rearrangement in the way we write the expression inside the limit symbol.

Examples:

1. Find $\lim_{x\to 0} 3x^2 \csc^2 x$.

   We need to rewrite into the form $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$ first.

$$\lim_{x\to 0} 3x^2 \csc^2 x = \lim_{x\to 0} 3x^2 \cdot \frac{1}{\sin^2 x} = \lim_{x\to 0} \frac{3x^2}{\sin^2 x}.$$

   Now we can use L'Hopital's rule.

$$\lim_{x\to 0} \frac{3x^2}{\sin^2 x} = \lim_{x\to 0} \frac{(3x^2)'}{(\sin^2 x)'} = \lim_{x\to 0} \frac{6x}{2\sin x \cos x}.$$

   Using L'Hopital's rule again, we obtain

$$\lim_{x\to 0} \frac{6x}{2\sin x \cos x} = \lim_{x\to 0} \frac{6}{2\sin x(-\sin x) + 2\cos x \cos x} = \lim_{x\to 0} \frac{6}{2(\cos^2 x - \sin^2 x)} = \frac{6}{2} = 3.$$

   Hence $\lim_{x\to 0} 3x^2 \csc^2 x = 3$.

2. Find $\lim_{y\to 0} \left(\csc y - \frac{1}{y}\right)$.

   This limit is of the form $\infty - \infty$. We can rearrange it first and then use L'Hopital's rule, as follows.

$$\lim_{y\to 0} \left(\csc y - \frac{1}{y}\right) = \lim_{y\to 0} \left(\frac{1}{\sin y} - \frac{1}{y}\right) = \lim_{y\to 0} \left(\frac{y - \sin y}{\sin y \cdot y}\right).$$

   Now, in the form $\frac{0}{0}$, we can use L'Hopital's rule and get

$$\begin{aligned} \lim_{y\to 0} \frac{y - \sin y}{y \sin y} &= \lim_{y\to 0} \frac{(y - \sin y)'}{(y \sin y)'} = \lim_{y\to 0} \frac{1 - \cos y}{\sin y + y \cos y} \\ &= \lim_{y\to 0} \frac{(1 - \cos y)'}{(\sin y + y \cos y)'} = \lim_{y\to 0} \frac{\sin y}{\cos y + \cos y + y(-\sin y)} \\ &= \frac{0}{1 + 1 - 0} = 0 \end{aligned}$$

We observe that usually the rearrangement of the expression so that it fits the condition of L'Hopital's rule is not difficult. For expressions of the form $\infty - \infty$, most of the times we can rearrange them properly by writing everything using the least common denominator (it will then become of the form $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$.

Exercises

1. $\lim_{x\to 0} \frac{2x-\sin x}{x} =$

   (a) 1 (b) 0 (c) 2 (d) ∞

2. $\lim_{x\to 0} \frac{x^3-3x^2+x}{x^3-2x} =$

   (a) 1 (b) -1 (c) 0 (d) ∞

3. $\lim_{x\to 0^-} \frac{3\sin x}{\sqrt{-x}} =$

   (a) -2 (b) $-\frac{1}{2}$ (c) 0 (d) 1

4. $\lim_{x\to 0} \frac{\tan x - x}{\sin 2x - 2x} =$

   (a) $-\frac{1}{4}$ (b) 0 (c)∞ (d) none of the answers

5. $\lim_{x\to 0} \frac{\sin x - \tan x}{x^2 \sin x} =$

   (a) 0 (b) $\frac{1}{2}$ (c)∞ (d) $-\frac{1}{2}$

6. $\lim_{x\to 0} \frac{2\csc^2 x}{\cot^2 x} =$

   (a) 1 (b) 2 (c) 0 (d) none of the answers

7. $\lim_{x\to \frac{\pi}{2}} \tan x - \sec x =$

   (a) 1 (b) $-\infty$ (c) ∞ (d) none of the answers

8. $\lim_{x\to 0} \csc^2 x - \cot^2 x =$

   (a) 0 (b) 1 (c) ∞ (d) none of the answers

9. $\lim_{xrightarrow\frac{1}{2}^-} \frac{(4-8x)^2}{\tan \pi x} =$

   (a) 0 (b) ∞ (c) 1 (d) none of the answers

10. $\lim_{x\to 0^+} \frac{x^2}{\sin x - x} =$

    (a) 0 (b) ∞ (c) $-\infty$ (d) does not exist

11. $\lim_{x\to \infty} x^2 - x =$

    (a) 0 (b) ∞ (c) 2 (d) does not exist

Solutions

1(a) 2(a) 3(c) 4(a) 5(d) 6(b) 7(d - the answer is 0) 8(b) 9(a) 10(c) 11(b)