LESSON 14: DERIVATIVES OF TRIGONOMETRIC AND INVERSE FUNCTIONS

Keywords
$(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$ (where $f(x_0) = y_0$)

Derivatives of Trigonometric Functions

We basically need to know that the derivative of the sine function is the cosine function, and the derivative of the cosine is the minus sine function. The derivatives of the other trigonometric functions can be derived from these two by the use of product rule and quotient rule.

To Show That $(\sin)'(x) = \cos x$

Here we need to recall the identity $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$, or $\sin(x+h) = \sin(x)\cos(h) + \sin(h)\cos(x)$.

$$
\begin{aligned}
(\sin)'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin x}{h} \\
&= \lim_{h\to 0} \left(-\sin x \frac{1-\cos h}{h} + \cos x \frac{\sin h}{h} \right) \\
&= (-\sin x) \left[\lim_{h\to 0} \frac{1-\cos h}{h} \right] + (\cos x) \left[\lim_{h\to 0} \frac{\sin h}{h} \right] \\
&= (-\sin x)\cdot 0 + (\cos x)\cdot 1 = \cos x
\end{aligned}
$$

The last line follows from the previous one because $\lim\limits_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim\limits_{h\to 0} \frac{1-\cos h}{h} = 0$.

To Show That $(\cos)'(x) = -\sin x$

Here we need to recall the trigonometric identity $\cos(x+h) = \cos x \cos h - \sin x \sin h$.

$$
\begin{aligned}
(\cos)'(x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0} \left(-\cos x \frac{1-\cos h}{h} - \sin x \frac{\sin h}{h} \right) \\
&= (-\cos x)\cdot 0 - (\sin x)\cdot 1 = -\sin x
\end{aligned}
$$

Again, to obtain the last line, we used the two very important and useful limits $\lim\limits_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim\limits_{h\to 0} \frac{1-\cos h}{h} = 0$.

Examples:

1. Find the the derivative of $3\cos x - 2\sin x$.

$$
\begin{aligned}
(3\cos x - 2\sin x)' &= 3(\cos x)' - 2(\sin x)' \\
&= 3(-\sin x) - 2\cos x = -3\sin x + 2\cos x
\end{aligned}
$$

2. Find the derivative of $\tan x$.

$$\begin{aligned}(\tan x)' &= \left(\frac{\sin x}{\cos x}\right)' \\ &= \frac{\cos x(\sin x)' - \sin x(\cos x)'}{(\cos x)^2} \\ &= \frac{\cos x \cos x - \sin x(-\sin x)}{(\cos x)^2} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = (\sec x)^2\end{aligned}$$

For the above, recall that $\cos^2 x + \sin^2 x = 1$.

3. Find the derivative of $\sin(2x)$.

For this one, we need to use the chain rule to get

$$(\sin 2x)' = \cos(2x) \cdot (2x)' = (\cos 2x) \cdot 2.$$

Using the rules for derivative, we find that

- $(\tan x)' = \sec^2 x$
- $(\sec x)' = \sec x \tan x$
- $(\cot x)' = -\csc^2 x$
- $(\csc x)' = -\csc x \cot x$

The above can all be derived from the derivatives of sine and cosine by use of the product rule and quotient rule, so what one really needs to memorize (if anything) is $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$.

Derivatives of Inverse Functions

Let's consider what happens to a line l_1 when it is reflected with respect to the line $y = x$, originating the line l_2. The slope m_2 of l_2 is equal to $\frac{1}{m_1}$, where m_1 is the slope of l_1 (provided that $m_1 \neq 0$, that is, l_1 is not horizontal).

Notice also that the point (x_0, y_0) when reflected with respect to the line $y = x$ gives the point (y_0, x_0).

Let f be a function that has an inverse function. The tangent line l_1 of f at a point (x_0, y_0) (here we assume $y_0 = f(x_0)$), when reflected with respect to the line $y = x$, gives the tangent line l_2 of f^{-1} at point (y_0, x_0), which is the point $(y_0, f^{-1}(y_0))$.

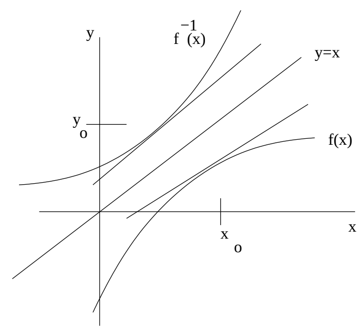
By our previous observations, we know that the slope of l_2 is the inverse of the slope of l_1. Adding all these information up, we get w_2 = slope of l_1 = derivative of f^{-1} at $y_0 = \frac{1}{m_1}$, where m_1 = slope of l_1 = derivative of f at x_0.

Therefore,

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)},$$

where $y_0 = f(x_0)$.

What is the use of the above formula? Sometimes we have the expression for f, we know that f has an inverse f^{-1}, and we need to find the derivative of f^{-1} at a certain point, but we cannot invert the expression for f (or we don't want to). Without inverting f, we can then use the relationship $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$, where $y_0 = f(x_0)$.



$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$

Examples

1. Let $y = f(x) = x^3 + 5x + 1$. What is $(f^{-1})'(7)$?

   We will assume f is invertible. We will see how to show that f has an inverse in the lesson on *Monotonicity and Concavity* (one can try plotting the graph of f and noticing that it is indeed invertible).

   By inspection, we see that $f(1) = 7$. We know that $(f^{-1})'(7) = \frac{1}{f'(1)}$, since $f(1) = 7$.

   Hence
   $$(f^{-1})'(7) = \frac{1}{f'(1)} = \frac{1}{3(1)^2 - 5 \cdot 1} = \frac{1}{8}.$$
   (Recall that $f'(x) = 3x^2 + 5$, therefore $f'(1) = 8$.)

2. If $f(x) = \cos x$ on $[0, \pi]$, what is the derivative f^{-1} at $\frac{\sqrt{2}}{2}$?

   We see that $f(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$. What is $(f^{-1})'(\frac{\sqrt{2}}{2})$?

   We have
   $$(f^{-1})'(\frac{\sqrt{2}}{2}) = \frac{1}{f'(\frac{\pi}{4})} = \frac{1}{(\cos)'(\frac{\pi}{4})} = \frac{1}{-\sin\frac{\pi}{4}} = \frac{1}{-\frac{\sqrt{2}}{2}} = -\sqrt{2}.$$

   Notice that the cosine function is indeed invertible on the interval $[0, \pi]$. Its inverse is called the arccosine function.

Exercises

1. The derivative of $f(x) = \frac{\sin 3x}{\cos 2x + 1}$ is

   (a) $\frac{3\cos 3x(\cos 2x+1) - 2\sin 2x \sin 3x}{(\cos 2x+1)^2}$ (b) $\frac{3\cos 3x}{-2\sin 2x+1}$ (c) $\frac{\cos 3x}{-\sin 2x}$ (d) $\frac{\cos 3x(\cos 2x+1) - \sin 2x \sin 3x}{(\cos 2x+1)^2}$

2. The derivative of $x^2 \tan x$ is

   (a) $2x \sec^2 x$ (b) $2x \tan x + x^2 \sec^2 x$ (c) $x^2 \sec^2 x$ (d) none of the answers

3. The derivative of $\sin^4(x^3 + 5x)$ is

   (a) $4\cos^3(x^3 + 5x) \cdot (3x^2 + 5)$
   (b) $4\sin^3(x^3 + 5x) \cdot \cos(x^3 + 5x) \cdot (3x^2 + 5)$
   (c) $\cos^4(3x^2 + 5)$

(d) $\cos^4(x^3+5x)$

4. The derivative of $\cos(\cos(\cos t))$ is

   (a) $-\sin(\cos(\cos t)) \cdot \sin(\cos t) \cdot \sin t$

   (b) $-\sin(\sin(\sin t))$

   (c) $\sin(\sin(\sin t))$

   (d) none of the answers

5. What is the equation of the tangent line to the graph of $y = 3\sin 2x$ at the point $(\frac{\pi}{2}, 0)$?

   (a) $y = x - \frac{\pi}{2}$ (b) $y = 6\cos 2x$ (c) $y - \frac{\pi}{2} = -6x$ (d) $y = -6\left(x - \frac{\pi}{2}\right)$

6. If $f(x) = 3x^5 + x - 2$, what is $(f^{-1})'(3)$?

   (a) $\frac{1}{241}$ (b) 16 (c) $\frac{1}{8}$ (d) $\frac{1}{16}$

7. If $g(y) = \sqrt{y+1}$, what is $(g^{-1})'(2)$?

   (a) $\frac{1}{\sqrt{3}}$ (b) $2\sqrt{2}$ (c) $2\sqrt{3}$ (d) $\frac{1}{2\sqrt{2}}$

Solutions

1(a) 2(b) 3(b) 4(a) 5(d) 6(d) 7(b)