LESSON 13: RULES FOR FINDING DERIVATIVES

Keywords
product rule, quotient rule, chain rule

To make the process of finding derivatives easier, there are a few properties we can use.

1. If $f(x) = k$ where k is a constant, then for any x, $f'(x) = 0$.
2. If $f(x) = x$, then $f'(x) = 1$, for any x.
3. If $f(x) = x^n$, where n is a positive integer, then $f'(x) = nx^{n-1}$.
   This can be shown by
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + nxh^{n-1} + h^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + h^n}{h} \\ &= \lim_{h\to 0} nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \ldots + h^{n-1} = nx^{n-1} \end{aligned}$$
4. If k is a constant and f is a differentiable function, then $(kf)'(x) = kf'(x)$. (this is the property of "pulling out the multiplying constants" while taking the derivative)
5. If f and g are differentiable functions, then $f+g$ and $f-g$ are differentiable functions, with $(f+g)'(x) = f'(x) + g'(x)$ and $(f-g)'(x) = f'(x) - g'(x)$.
6. If f and g are differentiable functions, then so is their product fg, and $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.
   This is an important and non-trivial rule, called the *product rule.* The derivative of the product of two functions is the derivative of the first times the second plus the first times the derivative of the second. IT IS NOT TRUE that the derivative of the product is the product of the derivatives, though this assumption is a mistake that people that just started learning calculus tend to make.
   To show that $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$:
$$\begin{aligned} (fg)'(x) &= \lim_{h\to 0} \frac{(fg)(x+h)-(fg)(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} f(x+h) \cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x) \cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= f(x)g'(x) + g(x)f'(x) \end{aligned}$$
   The passage to the last line of the proof was due to the definitions of derivative and also to the fact that, since f and g are continuous functions, $f(x+h) \to f(x)$ and $g(x+h) \to g(x)$ when $h \to 0$.

7. *Quotient Rule* If f and g are two differentiable functions with $g(x) \neq 0$, then their quotient $\frac{f}{g}$ is a differentiable function, and

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

We are not going to show how we obtain this formula, but there is a way to memorize this, which is by chanting " derivative of a quotient is : bottom square at the bottom (so we get g(x), the "bottom", squared at the bottom of our derivative), and on the top is bottom times derivative of the top minus top times derivative of the bottom." The full memorization of this "chant" without mistake only comes with time and practice... One should be careful about which of the terms in the numerator comes after the negative sign, since switching the order of the terms in the numerator will switch the derivative of the quotient by a sign! (note: others prefer the rhyme "lo dee hi minus hi dee lo" for the numerator of the derivative of the quotient, where "lo" stands for the denominator, "dee" stands for derivative and "hi" stands for the numerator)

8. *Chain Rule* Let $y = f(u)$ and $u = g(x)$. If g is differentiable at x and f is differentiable at $u = g(x)$, then the composite function $f \circ g$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

Careful that $(f \circ g)'(x) \neq f'(x)g'(x)$.

We shall see some applications of the chain rule to get an idea of how it works.

Examples:

1. Let $f(x) = x^5 - 34x^3 + 27$. Then $f'(x) = 5x^4 - 34 \cdot 3x^2 = 5x^4 - 102x^2$.
2. Let $f(x) = 10000000000000$. Then $f'(x) = 0$.
3. Let $f(x) = (3x^2 - 5)(5x^4 - 2x)$. Then, instead of expanding the parentheses and then computing the derivative of f, we can use the product rule.

$$f'(x) = (6x)(5x^4 - 2x) + (3x^2 - 5)(20x^3 - 2) = 30x^5 - 12x^2 + 60x^5 - 6x^2 - 100x^3 + 10.$$

4. Let $f(x) = \frac{\sqrt{x}}{x^2 - x + 1}$. Then

$$f'(x) = \frac{(x^2 - x + 1)\frac{1}{2\sqrt{x}} - \sqrt{x}(2x - 1)}{(x^2 - x + 1)^2},$$

by the quotient rule.

5. Let $f(x) = x^{-n}$, where n is a positive number. Then

$$f'(x) = \left(\frac{1}{x^n}\right)' = \frac{x^n \cdot 0 - 1 \cdot nx^{n-1}}{x^{2n}} = \frac{-nx^{n-1}}{x^{2n}} = -nx^{-n-1}.$$

6. Let $f(x) = (3x^2 - x)^{54}$. Then $f'(x) = 54(3x^2 - x)^{53}(6x - 1)$.

To show this, consider the functions $g(x) = x^{54}$ and $h(x) = 3x^2 - x$. Then $f = g \circ h$, and $f'(x) = g'(h(x)) \cdot h'(x)$. Since $g'(x) = 54x^{53}$, we have $g'(h(x)) = 54(3x^2 - x)^{53}$. As $h'(x) = 6x - 1$, we get $f'(x) = 54(3x^2 - x)^{53}(6x - 1)$.

7. Let $f(x) = [(3x^4 - x^2)(x^5 - 2x^3 + 1)]^{-40}$. Then $f'(x) = -40[(3x^4 - x^2)(x^5 - 2x^3 + 1)]^{-41} \cdot [(12x^3 - 2x)(x^5 - 2x^3 + 1) + (3x^4 - x^2)(5x^4 - 6x)]$, by the chain rule, the product rule and Example 5 (which states that $(x^{-n})' = -nx^{-n-1}$, when n is a positive integer).

Exercises

1. The derivative of $f(x) = 2x^2$ is
   (a) x^2 (b) 2 (c) $2x$ (d) $4x$
2. The derivative of $f(x) = x^4 + x^3 + x^2 + x + 1$ is
   (a) $x^3 + x^2 + x + 1$ (b) $12x$ (c) $4x^3 + 3x^2 + 2x + 1$ (d) $4x^3$
3. The derivative of $h(t) = t^{-4} + \frac{3}{t^3}$ is
   (a) $-4t^{-5} + \frac{1}{t^2}$ (b) $-4t^{-5} - \frac{9}{t^4}$ (c) $-4t^{-3} - \frac{9}{t^2}$ (d) $-4t^{-3} + \frac{1}{t^2}$
4. The derivative of $p(y) = \frac{y-1}{y+1}$ is
   (a) 1 (b) $\frac{-2y}{(y+1)^2}$ (c) $\frac{2}{(y+1)^2}$ (d) $\frac{2y}{(y+1)^2}$
5. The derivative of $g(x) = (x^3 - 2x + 1)^{100}$ is
   (a) $100(x^3-2x+1)^{99}$ (b) $100(x^3-2x+1)^{101}$ (c) $100(x^3-2x+1)^{99}(3x^2-2)$ (d) $100(3x^2-2)^{99}$
6. The derivative of $y(x) = (5x^2 - 7)(3x^2 - 2x + 1)$ is $y'(x) = 10x(6x - 2)$
   (a) True (b) False
7. If $f(0) = 4$, $f'(0) = -1$, $g(0) = 2$, and $g'(0) = -5$, then $(fg)'(o) =$
   (a) 5 (b) -22 (c) 11 (d) 13
8. If $f(0) = 4$, $f'(0) = -1$, $g(0) = 2$, and $g'(0) = -5$, then $(f + g)'(0) =$
   (a) 8 (b) 6 (c) -14 (d) -6
9. If $f(0) = 4$, $f'(0) = -1$, $g(0) = 2$, and $g'(0) = -5$, then $\left(\frac{f}{g}\right)'(0) =$
   (a) $\frac{9}{2}$ (b) $\frac{1}{5}$ (c) $\frac{-1}{22}$ (d) $\frac{-1}{2}$
10. The equation of the tangent line to $y = \frac{1}{x^2+1}$ at the point $(1, \frac{1}{2})$ is
    (a) $y = \frac{-2x}{(x^2+1)^2}$ (b) $y = x + \frac{1}{2}$ (c) $y = \frac{1}{2}x$ (d) $y = 1 - \frac{1}{2}x$
11. Find the derivative of $f(x) = \frac{(x+1)^2}{3x-4}$.
12. Find the derivative of $g(x) = (2 - 3x^2)^4(x^7 + 3)^3$.
13. Find the derivative of $h(x) = \left(\frac{x-2}{x-\pi}\right)^{-3}$.

Solutions

1(d) 2(c) 3(b) 4(c) 5(c) 6(b) (the derivative is $10x(3x^2-2x+1)+(5x^2-7)(6x-2)$ by the product rule) 7(b) 8(d) 9(a) 10(d) 11(the derivative is $\frac{2(x+1)\cdot(3x-4)-3\cdot(x+1)^2}{(3x-4)^2} = \frac{(x+1)(3x-11)}{(3x-4)^2}$ by the quotient rule) 12(the derivative is $4(2-3x^2)^3(-6x)(x^7+3)^3 + (2-3x^2)^4 3(x^7+3)^2(7x^6)$ by a combination of product rule with chain rule) 13(the derivative is $-3\left(\frac{x-2}{x-\pi}\right)^{-4} \cdot \left(\frac{2-\pi}{(x-\pi)^2}\right)$ by a combination of chain rule and quotient rule)