LESSON 12: DERIVATIVE

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derivative, differentiable function, differentiation, average velocity, instantaneous velocity, tangent line

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The *derivative* of a function is basically the *instantaneous rate of change* of the function.

Think of yourself driving a car. You travelled 45 miles within three hours. So, your average velocity was 15 miles per hour. Unless you can keep your speed pretty constant, it is unlikely that the speedometer of the car was always indicating 15 miles/hour. What your speedometer shows is the *instantaneous velocity* of the car.

Suppose the function $s(t) = 5t^2$ denotes the space s that the car is occupying, depending on the time t given in hours. At time $t = 0$, the car was at the origin ($s = 0$). After three hours, the car is at $s(3) = 5 \cdot 3^2 = 45$. So you've travelled 45 miles in three hours. Does that mean your velocity was always 15 miles/hour? We will see that with $s(t) = 5t^2$, this won't be true.

What was the speedometer of the car showing when you have travelled exactly an hour (that is, when $t = 1$)?

To compute this, we can think of it this way. Compute the average velocity for small intervals of time around $t = 1$ and then *take the limit* when the size of the intervals of time go to zero. That would give us the velocity at exactly $t = 1$.

In other words, let's compute:

$$\lim_{\Delta t \to 0} \frac{s(1 + \Delta t) - s(1)}{\Delta t},$$

where Δt denotes a small interval of time (positive or negative).

We get

$$\begin{aligned}
\lim_{\Delta t \to 0} \frac{s(1 + \Delta t) - s(1)}{\Delta t} &= \lim_{\Delta t \to 0} \frac{5(\Delta t + 1)^2 - 5}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{5(\Delta t^2 + 2\Delta t + 1) - 5}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{5\Delta t^2 + 10\Delta t + 5 - 5}{\Delta t} = \lim_{\Delta t \to 0} \frac{5\Delta t^2 + 10\Delta t}{\Delta t} \\
&= \lim_{\Delta t \to 0} (5\Delta t + 10) = 10.
\end{aligned}$$

So the instantaneous velocity of the car at time $t = 1$ hour is 10 miles/hour, which is different from the average velocity, 15 miles/hour. We can guess that the car accelerated after an hour, otherwise the average velocity in three hours wouldn't be greater than the velocity at $t = 1$.

Here is the definition of *derivative*:

The *derivative* of a function f is another function f' (read "f prime") whose value at any number c is given by

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

provided that the above limit exists and is finite (that is, it can't be $-\infty$ or ∞).

If the limit does exist, we say that f is *differentiable* at c. Finding a derivative is called *differentiation*.

Examples:

1. Let $f(x) = 4x - 2$. Find $f'(4)$.

$$\begin{aligned} f'(4) &= \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} \\ &= \lim_{h\to 0} \frac{4(4+h) - 2 - (4\cdot 4 - 2)}{h} \\ &= \lim_{h\to 0} \frac{16 + 4h - 2 - 16 + 2}{h} \\ &= \lim_{h\to 0} \frac{4h}{h} = \lim_{h\to 0} 4 = 4 \end{aligned}$$

2. Let $f(x) = 2x^3 - 1$. Find $f'(c)$.

$$\begin{aligned} f'(c) &= \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \\ &= \lim_{h\to 0} \frac{2(c+h)^3 - 1 - (2\cdot c^3 - 1)}{h} \\ &= \lim_{h\to 0} \frac{2(c^3 + 3c^2h + 3ch^2 + h^3) - 1 + 2c^3 + 1}{h} \\ &= \lim_{h\to 0} \frac{2c^3 + 6c^2h + 6ch^2 + 2h^3 - 1 - 2c^3 + 1}{h} \\ &= \lim_{h\to 0} \frac{6c^2h + 6ch^2 + 2h^3}{h} \\ &= \lim_{h\to 0} (6c^2 + 6ch + 2h^2) = 6c^2 \end{aligned}$$

3. Let $f(x) = \frac{1}{x}$. Find $f'(x)$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h\to 0} \left[\frac{x - (x+h)}{(x+h)x} \cdot \frac{1}{h}\right] \\ &= \lim_{h\to 0} \left[\frac{-h}{(x+h)x} \cdot \frac{1}{h}\right] \\ &= \lim_{h\to 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2} \end{aligned}$$

4. Let $k(x) = \sqrt{x}$. Find $k'(x)$.

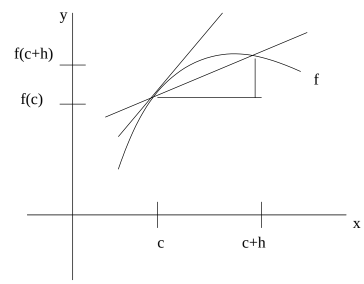
   We will need a little common trick to do this computation...

$$\begin{aligned} k'(x) &= \lim_{h\to 0} \frac{k(x+h) - k(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h\to 0} \left[\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right] \end{aligned}$$

$$\begin{aligned} &= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x}} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x}} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

Graphical interpretation of the derivative

Graphically, the value $f'(c)$ corresponds to the slope of the tangent line to the graph of f at $x = c$.



What is the equation of the tangent line to $y = x^2 - 2x + 2$ at the point $(1, 1)$?

To solve this, we first notice that the slope of the tangent line to y at $(3, 5)$ is the value of the derivative of $x^2 - 2x + 2$ at $x = 3$. Since the derivative of $x^2 - 2x + 2$ is $2x - 2$, at $x = 3$ it has value 4. Therefore, the tangent line goes through the point $(3, 5)$ and has slope 4. Its equation is $y - 5 = 4(x - 3)$ or $y = 4x - 7$.

Equivalent Forms for the Derivative

We do not have to use the letter h in the definition of $f'(c)$. The two limits $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ and $\lim_{s\to 0} \frac{f(c+s)-f(c)}{s}$ are the same and they both mean $f'(c)$.

Another way of writing $f'(c)$ is

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c}.$$

We obtain the above formula if we think of h to be equal to $x - c$. Then, when $h \to 0$, we have $x \to c$, and $f(c + h) = f(x)$, and we rewrite $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ as $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$.

Depending on the situation, using one form or the other for the derivative might be more useful.

Differentiability Implies Continuity

If a function f is differentiable at a point c, then the function is continuous at c. Equivalently, for the graph of a function to have a tangent line at a certain point c, the graph needs to "behave nicely" near the point c. In other words, if $f'(c)$ exists, then f is continuous at c.

To show this, we need to show that $\lim_{x\to c} f(x) = f(c)$ (this is the definition of continuity of f at c). We begin by writing $f(x)$ in a fancy way.

$$f(x) = f(c) + \frac{f(x)-f(c)}{x-c} \cdot (x-c), \quad x \neq c$$

Then

$$\begin{aligned}\lim_{x\to c} f(x) &= \lim_{x\to c}\left[f(c) + \frac{f(x)-f(c)}{x-c}\cdot(x-c)\right]\\ &= \lim_{x\to c} f(c) + \lim_{x\to c}\frac{f(x)-f(c)}{x-c}\cdot\lim_{x\to c}(x-c)\\ &= f(c)+f'(c)\cdot 0\\ &= f(c)\end{aligned}$$

It is **FALSE** to say that if a function f is continuous at c, then it must be differentiable at c. Consider the example $f(x)=|x|$, the absolute value function. It is continuous at all real numbers, in particular it is continuous at the $x=0$. But f does not have a derivative at 0, as we shall see.

Notice that $\frac{f(0+h)-f(0)}{h}=\frac{|0+h|-|0|}{h}=\frac{|h|}{h}$. If $h<0$, then $|h|=-h$, and $\frac{|h|}{h}=-1$. If $h>0$, then $|h|=h$, and $\frac{|h|}{h}=1$.

Hence $\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=-1$ and $\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=1$. Since the right hand limit and the left hand limit don't match, f does not have a derivative at 0.

The fact that the absolute value function does not have a derivative at 0 can also be seen graphically. We can see that the graph of the absolute value function has a 'cusp' at $x=0$, a sudden sharp curve. What would be the tangent line to the graph at $x=0$? If we look only to the left of 0, we think that the tangent line should be the line $y=-x$, with slope -1. On the other hand, looking only to the right of 0, we think that the tangent line should be $y=x$, with slope 1. Therefore, there is not actually any tangent line to the graph at zero.

Usually, when the graph of a function f has a very sharp curve or corner that doesn't look smooth at a certain point c or if the graph of f has a vertical tangent at c, then f will not have a derivative at c.

Of course, if f is not continuous at c, then f will not be differentiable at c. Remember always that f *needs* to be continuous at c for it to be differentiable at c; but just because f is continuous *won't* necessary *imply* that it is going to be differentiable.

Higher Order Derivatives

There is no mystery in higher order derivatives. The process of differentiation takes a function f and produces a new function f', called the (first) derivative of f. If we now differentiate f', we get another function, f'', which is called the *second derivative* of f. Continuing in this manner, we can obtain the third, fourth, n-th derivative of f.

In Physics, if $s(t)$ is a function that indicates the location/space of a certain object at time t, then the first derivative of s, $s'(t)$, is the velocity of the object at time t, and the second derivative of s, $s''(t)$, is the acceleration of the referred object.

Example:

If $f(x)=x^3-5x^2+7x-12$, then

$$\begin{aligned}f'(x) &= 3x^2-10x+7\\ f''(x) &= 6x-10\\ f'''(x) &= 6\\ f^{iv}(x) &= f^{v}(x)=f^{vi}(x)=\ldots=f^{(n)}(x)=0 \quad (n\geq 4)\end{aligned}$$

Exercises

1. If $f(x) = -2x + 1$, then $f'(x) =$
   (a) -2 (b) 1 (c) 0 (d) $-2x$
2. If $h(x) = \frac{2}{x}$, then $h'(x) =$
   (a) $-\frac{1}{x^2}$ (b) $\frac{2}{x}$ (c) $-\frac{2}{x^2}$ (d) $\frac{2}{x^2}$
3. If $f(x) = x^2$, then $f'(0) =$
   (a) $2x$ (b) 0 (c) 2 (d) 1
4. If $g(x) = x^2 - x$, then $g'(1) =$
   (a) 1 (g) 0 (c) 2 (d) -1
5. If $k(x) = \sqrt{5x}$, then $k'(x) =$
   (a) $\sqrt{5x}$ (b)$-\frac{1}{\sqrt{5x}}$ (c) $-\frac{1}{5x}$ (d) $\frac{\sqrt{5}}{2\sqrt{x}}$
6. The limit $\lim_{x\to 2} \frac{x^2-4}{x-2}$ is a derivative, but of what function at what point?
   (a) of x^2 at 4 (b) of x^2 at 2 (c) of x at 2 (d) of $x^2 - 4$ at 0
7. The limit $\lim_{t\to x} \frac{t^3-x^3}{t-x}$ is a derivative, but of what function at what point?
   (a) of t at x (b) of $t^3 - 1$ at 1 (c) of t^3 at x (d) none of the previous options
8. If $g(x) = x^3 + 3x^2 + 6x$, then $g'''(x) =$
   (a) $3x^2 + 6x + 6$ (b)6 (c) $6x + 6$ (d) 0
9. If $h(t) = \frac{t}{1-t}$, then the fourth derivative of h is equal to (*come back to this exercise after Lesson 13*)
   (a) $\frac{6}{(1-t)^4}$ (b)$\frac{1}{(1-t)^2}$ (c) $\frac{24}{(1-t)^5}$ (d) $\frac{2}{(1-t)^3}$
10. If $f(x) = 5x^3 + 2x^2 - x$, then $f''(2) =$
    (a) 42 (b) 67 (c) 30 (d) 64
11. If $h(y) = \frac{2y^2}{5-y}$, then $h''(0) =$ (*come back to this exercise after Lesson 13*)
    (a) $\frac{4}{5}$ (b) 0 (c) $\frac{28}{25}$ (d) none of the other options
12. If a function is continuous on an interval (a, b) containing c, then it is differentiable at c.
    (a) True (b) False

Solutions

1(a) 2(c) 3(b) 4(a) 5(d) 6(b) 7(c) 8(b) 9(c) 10(d) 11(a) 12(b)