LESSON 11: SOME PROPERTIES OF CONTINUOUS FUNCTIONS
or
THE INTERMEDIATE VALUE THEOREM

Keywords

Intermediate Value Theorem, maximum value, minimum value

Throughout our lessons, we have been trying to avoid the use of the word "theorem". In this lesson, the *intermediate value theorem* is such an important statement that it has to be called a theorem. We will see other statements concerning continuous functions that, unfortunately, don't have a nice and famous name like the intermediate value property.

The intermediate value theorem states that:

If f is a continuous function defined on $[a, b]$ and W is a number between $f(a)$ and $f(b)$, then there must be at least one number c lying between a and b such that $f(c) = W$.

The first thing we notice is the requirement that f be continuous throughout the whole interval $[a, b]$ (it is definitely not enough to have f continuous only at a and at b). The second thing is, there might be more than one point c inside $[a, b]$ such that $f(c) = W$.

Examples:

1. Consider the function $f(x) = x^4 - 5x + 2$ on the interval $[0, 1]$. We know that, being a polynomial function, f is continuous on the interval $[0, 1]$. We can easily see that $f(0) = 2$ and $f(1) = -2$. Zero is a number that lies between 2 (which is $f(0)$) and -2 (which is $f(1)$). Hence, by the intermediate value theorem, there is a number c that lies between 0 and 1 such that $f(c) = 0$.

   This is one of the most useful applications of the theorem, since it is sometimes necessary to know the zeroes of a polynomial function. To try to solve the equation $f(x) = x^4 - 5x + 2 = 0$ might not be easy, but with the intermediate value theorem we know that there is a zero to the function at a point between 0 and 1 !

2. Show that there is a number x such that $\cos x = x$.

   How do we do this one? Let's consider the function $g(x) = \cos x - x$. We know that g is a continuous function, we also know that $g(0) = \cos 0 - 0 = 1$ and $g(\frac{\pi}{2}) = \cos \pi 2 - \frac{\pi}{2} = 0 - \frac{\pi}{2} = -\frac{\pi}{2}$. Since 0 is a value between 1 and $-\frac{\pi}{2}$, by the intermediate value theorem there must be at least a number c between 0 and $\frac{\pi}{2}$ such that $g(c) = 0$. Hence $g(c) = \cos c - c = 0$ implies $\cos c = c$.

3. Show that there is a number x such that $\frac{x-2}{x+5} = -\frac{1}{5}$.

   Consider the function $f(x) = \frac{x-2}{x+5}$ on the interval $[0, 2]$. It is continuous, and $f(0) = -\frac{2}{5}$, $f(2) = 0$. Since the value $-\frac{1}{5}$ is in between $-\frac{2}{5}$ and 0, by the intermediate value theorem there is a point x in the interval $(0, 2)$ such that $\frac{x-2}{x+5} = -\frac{1}{5}$ (even though we cannot say which number exactly is x!).

Graphically, the intermediate value theorem says that: if a function f is continuous on $[a, b]$, and $f(a) = m$ and $f(b) = M$; then every horizontal line between the horizontal lines $y = m$ and $y = M$ will cut the graph of f at *at least* one point between a and b. This is shown in Figure 1.

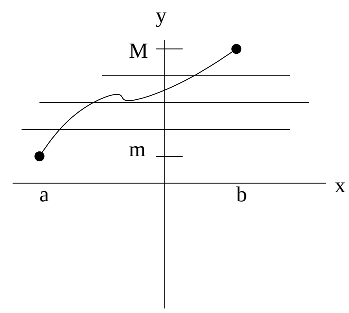
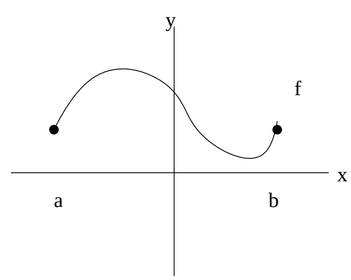
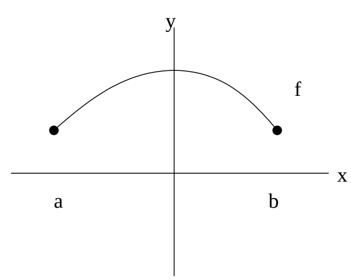


Figure 1

What if $f(a) = f(b)$? This *by no means* implies that f is constant between the points a and b! In the case where f takes the same value at the endpoints of the interval, the intermediate value theorem doesn't tell us much more than what we already know. The graph could look like



or like

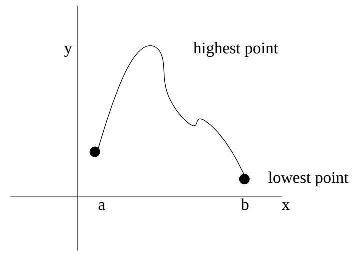


or something else.

Other Results On Continuous Functions

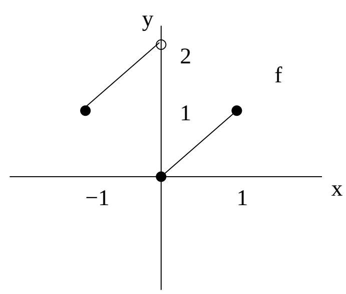
If f is a continuous function defined on the closed interval $[a, b]$, then f attains a maximun *and* a minimum (these two values could be the same) on the interval. In other words, if f is a continuous function on $[a, b]$, then there is a point c in $[a, b]$ such that $f(x) \leq f(c)$ for all x in $[a, b]$ (that is, the value $f(c)$ is the highest value taken by f in the interval $[a, b]$), and there is a point d in $[a, b]$ such that $f(x) \geq f(d)$ for all x in $[a, b]$. Notice that the points c and d might not be unique (check that they will be unique only if f is an injective function!).

Graphically, the above says that the graph of a continuous function defined on a closed interval reaches a "highest point" and also a "lowest point".



To convince ourselves that the condition f being continuous on the interval is necessary, consider the function f defined by

$$f(x) = \begin{cases} x+2 & \text{if } -1 \geq x < 0 \\ x & \text{if } 0 \leq x \leq 1 \end{cases}$$



Then f does not have a maximum value on the interval $[-1, 1]$.

To convince ourselves that, even if the function is continuous, the interval that we are considering *must be closed*, let's look at the example where $g(x) = \frac{1}{x}$ on $(0, \infty)$. It is clear that g does not have a maximum or a minimum value on the interval $(0, \infty)$.

Therefore, the "maximum/minimum value statement" depends on some intrinsic properties of the real numbers and of closed intervals. Unfortunately, we shall not discuss what properties are these.

One observation: we have a *guarantee* that if f is a continuous function on a closed interval, then f attains a maximum and a minimum value at that interval, but we might not be able to determine what these max/min values are. In further lessons we shall have some tools to find the maximum and minimum points using the derivative of f.

Exercises

1. Use the Intermediate Value Theorem to prove $x^3 + 3x - 2 = 0$ has a real solution between 0 and 1.

2. Let $f(x) = \frac{1}{x-1}$. Then $f(-2) = \frac{-1}{3}$ and $f(2) = 1$. The Intermediate Value Theorem implies the existence of a number c between -2 and 2 such that $f(c) = 0$.

   (a) True (b) False

3. Sketch the graph of a function f that satisfies all the following conditions

   (a) its domain is $[-3, 3]$

   (b) $f(-3) = f(-1) = f(1) = f(3) = 2$

(c) it is discontinuous at -1 and 3

(d) if is right continuous at -1 and left continuous at 1.

4. It a function f is not continuous on $[a, b]$, then it does not have a maximum nor a minimum value on $[a, b]$.

   (a) True (b) False

Solutions

1(consider the function $f(x) = x^3 + 3x - 2$ on the interval $[0, 1]$. We see that f is continuous, and $f(0) = -2$, $f(1) = 2$. Hence, by the Intermerdiate Value Theorem, there is a point c in $[0, 1]$ such that $f(c) = 0$, because 0 is a value between $f(0)$ and $f(1)$.) 2(b - f is not continuous on the interval $[-2, 2]$, since it is not defined for $x = 1$. The Intermediate Value Theorem does not apply in this case) 3 (solutions may vary) 4(b - even if f is not continuous, it *might* still have maximum and minimum values)