LESSON 10: CONTINUITY OF FUNCTIONS

Keywords
continuous function, discontinuous function, left continuous, right continuous, removable discontinuity, jump discontinuity, essential discontinuity

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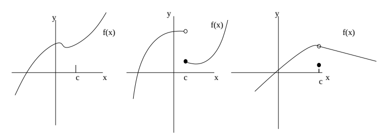
In mathematics, we use the word *continuous* to describe functions that do not have abrupt changes. For example, look at the following three graphs:



Among the three graphs above, only the first graph is considered to be "continuous" at c. In other words, a function f will be called continuous at c if for x near c the values of $f(x)$ are near $f(c)$, that is, in some sense the value of f at c can be "determined" by the values of f at points very close to c. To make this formal, we have:

Let f be defined at an open interval containing c. We say that f is *continuous at* c if

$$\lim_{x\to c} f(x) = f(c).$$

With this definition of continuous functions, there is an implicit requirement that : (1) $\lim_{x\to c} f(x)$ exists (and is finite): (2) $f(c)$ exists, that is, f is defined at c; and (3) $\lim_{x\to c} f(x) = f(c)$. If any of these requirements fails, then f is not continuous at c.

A function is called *continuous* if it is continuous at all points of its domain.

A function that is *not* continuous is called *discontinuous.* (we can also sometimes say that a function is discontinuous at a certain point c)

Or: the value of f at c is the value that we think it should be, the natural one. To make this clearer, there is a very simple way to recognize that a graph is the graph of a continuous function. If we can draw the graph of f *without lifting our hand from the paper*, then the function f is continuous. Look at the three graphs above and convince yourself that the first one is the only one we can draw without having to lift our pencil!

In other words, a continuous function has no "breaks" or "jumps". An example of a discontinuous function is the function that associates to each letter its corresponding postage. The postage machine charges a letter that weights one ounce or less \$0.33, but if the letter weights slightly more than an ounce - even it is just a little bit, as long as it can be detected by the weighing machine at the post office - it will be charged \$0.55. There is such a big jump on the postage value for small changes of letter-weight that the postage machine is a discontinuous function.

Take the function $f(x) = \frac{x^2-1}{x-1}$. Since this function is not defined at $x = 1$ (because $x - 1$ makes the denominator equal zero), f is not continuous at $x = 1$. Is there any way we can define f at 1 in order to make f continuous? Notice that $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} x + 1 = 2$. So, $\lim_{x\to 1} f(x) = 2$. If we define $f(1)$ to be equal 2, then f will be continuous at 2!

Let $g(x) = \frac{x^2-4}{x+2}$. Check that if we define $g(-2) = -4$, then g is continuous at -2.

The two examples of discontinuities shown above are easily removable with a good definition of the function at the point of discontinuity. Some people like to call this kind of discontinuity a *removable discontinuity.* Before you start thinking that ALL discontinuities are of the removable kind, remember the function h defined by $h(x) = \frac{1}{x}$. It has a vertical asymptote at $x = 0$ (see previous lesson) and there is no way we can remove such a discontinuity!

Continuity of Familiar Functions

Most functions that we will meet here are either continuous everywhere (on the real line) or continuous everywhere except at a few points.

A polynomial function is continuous everywhere (at all real numbers), a rational number is continuous everywhere it is defined (that is, everywhere except where the denominator is zero).

The absolute value function is continuous at every real number. If n is odd, the n-th root function is continuous at every real number. If n is even, the n-th root function is continuous at every non-negative real number (and it is not defined for negative numbers).

The sine and cosine functions are continuous at every real number, and the functions $\tan x$, $\cot x$, $\sec x$, $\csc x$ are continuous at every real number c in their domains. For example, $\tan x$ is not continuous at $x = \frac{\pi}{2}$, because we have $\cos(\frac{\pi}{2}) = 0$ (that is, $\frac{\pi}{2}$ is not in the domain of the tangent function).

If two functions, f and g, are continuous at a point c then:

- their sum $f + g$ is continuous at c;
- their difference $f - g$ is continuous at c;
- their product $f \cdot g$ is continuous at c;
- their quotient $\frac{f}{g}$ is continuous at c, *provided that* $g(c) \neq 0$;
- the function kf, where k is a constant number, is continuous at c.

If $\lim_{x \to c} g(x) = L$, and if f is continuous at L, then

$$\lim_{x \to c} f(g(x)) = f(\lim_{x \to c} g(x)) = f(L).$$

In particular, if g is continuous at c, and f is continuous at $g(c)$, then the composite function $f \circ g$ is continuous at c.

If f is continuous at $g(c)$, but g is not continuous at c, then $f \circ g$ *might not be* continuous at c ! To assume that we don't need the condition "g is continuous at c" is a very common mistake. Let's see an example: take $f(x) = x$ and

$$g(x) = \begin{cases} 0 & \text{if } x = c \\ 1 & \text{if } x \neq c. \end{cases}$$

Then

$$f(g(x)) = \begin{cases} 0 & \text{if } x = c \\ 1 & \text{if } x \neq c. \end{cases}$$

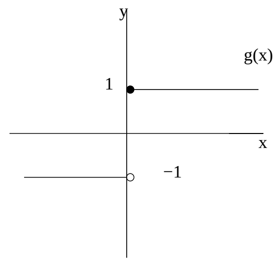
We see that f is continuous at $g(c)$, but $f \circ g$ is not continuous at c because g is not continuous at c.

Sometimes a function f is only defined in a closed interval of the type $[a, b]$, where a and b are real numbers. How can we say f is continuous at a, if we cannot talk about $\lim_{x \to a} f(x)$, only about $\lim_{x \to a^+} f(x)$ (since f is not defined for numbers smaller than a)? In these cases, we need to consider *right continuity* and *left continuity*.

The function f is *right continuous* at a if $\lim_{x \to a^+} f(x) = f(a)$ and *left continuous* at b if $\lim_{x \to b^-} f(x) = f(b)$.

We say that a function f is continuous on the interval $[a, b]$ if it is continuous at every point of the open interval (a, b) and it is also right continuous at a and left continuous at b.

When a function is left continuous and right continuous at a point c, but the left hand limit $\lim_{x \to c^-} f(x)$ and the right hand limit $\lim_{x \to c^+} f(x)$ are not the same, then some people say we have a



"jump discontinuity" at c. An example is the function g defined by $g(x) = 0$ if $x \geq 0$ and $g(x) = -1$ if $x < 0$. Then g has a *jump discontinuity* at 0.

When the discontinuity of a function f at a point c is not of the removable kind nor the jump kind, we say f was an *essential discontinuity* at c. Basically, having an essential discontinuity at c means that the function is really not nice at c. Take the function $f(x) = \frac{1}{x}$. The discontinuity of f at 0 is of the essential kind.

Exercises

1. The function $f(x) = (x-2)(x-4)$ is continuous at 2.
   (a) True (b) False
2. The function $h(x) = \frac{x}{x-2}$ is continuous at 2.
   (a) True (b) False
3. The function $g(x) = |x-2|$ is continuous at 2.
   (a) True (b) False
4. At what points, if any, is the function $T(\theta) = \tan\theta$ discontinuous?
   (a) the function is continuous everywhere
   (b) at $\theta = \frac{\pi}{4}$
   (c) at all points of the form $\theta = \frac{\pi}{2} + k\pi$, where k is an integer
   (d) at all points of the form $\theta = 2k\pi$, where k is an integer
5. From the graph of l given in Figure 1, indicate the intervals on which l is continuous.

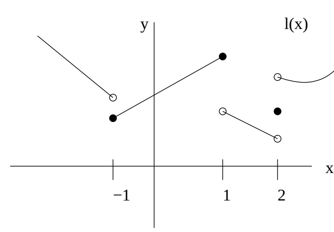


Figure 1

Solutions

1(a) 2(b) 3(a) 4(c) 5($(-\infty,-1) \cup [-1,1] \cup (1,2) \cup (2,\infty)$)