LESSON 1: FUNCTIONS

Keywords
function, domain, range, injective function, surjective function, bijective function, absolute value function

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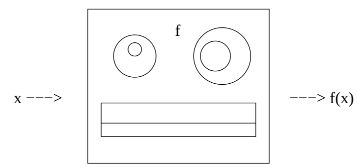
A *function* f is a rule that associates to each object x in a set (called *domain*) one single element $f(x)$ from a second set. The *range* is the set of all elements in this second set that have a corresponding x from the first set.

You can imagine the function f as a "machine" that, given an input x, produces an output $f(x)$.



The "machine" f

Examples of functions:

1. Let f be the function whose domain is the set of all possible English names such that $f(x) =$ first letter of the name. In other words, we have

   $f(Andrea) = a$, $f(Markus) = m$, and so on.

   Here, the range would be the letters of the alphabet (since, if we use our imagination, we can think of names that start with all the letters of the alphabet).

2. Let g be the function whose domain is the set of all real numbers such that $g(x) = 2x$. That is, we are defining g as the "machine" that takes whatever number you plug in and outputs twice the value of that number. In this case, we have $g(2) = 4$, $g(0.5) = 1$, etc. The range of g is the set of all real numbers (for every real number is twice the value of some other real number).

From now on, unless otherwise specified, the domain of any function shall be the set of all real numbers. In other words, we won't consider functions like Example 1 above.

Also notice that the choice of the letters "f" and "x" to denote an arbitrary function and its input is not mandatory. We can talk of a function g such that $g(x) = x - 2$ and of a function h such that $h(y) = y - 2$ and we would be referring to the *same* function, basically the "machine" that takes an input and subtracts 2 from it and give this resulting number as the output.

A function f is *injective* if, given $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$. In other words, two different inputs x_1 and x_2 never give the same output. Going back to Examples 1 and 2 above, we see that the function in Example 1 is not injective. Why? We have $Anna \neq Andrea$ but $f(Anna) = f(Andrea) = A$, so two different names give the same ouput. Thus f is not injective. On the other hand, in Example 2, if two numbers have the same doubled value, they must be the same; therefore, two different numbers will have different doubled values, which means g is an injective function.

A function f is *surjective* when every element y from the second set (the "target" set) is an output for some element x of the domain. In other words, given any y from the second set, we can find an x from the first set such that $f(x) = y$. In Example 1 above, f is a surjective function, because every letter of the alphabet starts some English name. Function g in Example 2 above is also a surjective function, because every real number y is going to be the double of some real number x, namely $y/2$, since $g(y/2) = 2(y/2) = y$!

A function f that is both injective and surjective is called *bijective*.

An important special function is the *absolute value function*, whose domain is the set of all real numbers, and is defined by

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x > 0 \end{cases}$$

The absolute value function always gives a non-negative number (that is, a number greater or equal than 0) as ouput and its range is the set of all nonnegative numbers. In the next section we will look at the graph of this function.

Exercises

1. For $f(x) = 1 - x^2$, find $f(-2)$ and $f(2)$, respectively.

   (a) -3 and -3 (b) 3 and -3 (c) -3 and 3 (d) 5 and -3

2. For $g(y) = \frac{1}{\sqrt{y-3}}$, find $g(3+\sqrt{2})$.

   (a) $\sqrt[3]{2}$ (b) $\frac{\sqrt{2}}{2}$ (c) $\frac{1}{\sqrt[4]{2}}$ (d) does not exist

3. The function f given by $f(x) = 1 - x^2$ is injective.

   (a) True (b) False

4. The function $h(x) = x^2$ is surjective.

   (a) True (b) False

5. For $G(z) = \frac{1}{z-1}$, find $G(0.999)$.

   (a) 0.001 (b) -0.001 (c) 1000 (d) -1000

6. For $F(x) = x^3 + 3x$, find $F(\sqrt{2})$.

   (a) $5\sqrt{2}$ (b) $4\sqrt{2}$ (c) $8 + 3\sqrt{2}$ (d) $4 + 3\sqrt{2}$

7. Which of the following determines a function f with formula $y = f(x)$? (Hint: Solve for y in terms of x and note that the definition of a function requires a single y for each x.)

   (a) $x^2 + y^2 = 4$ (b) $x = \sqrt{y+1}$ (c) $|xy| = 1$ (d) $\frac{x^2}{4} + \frac{y^2}{9} = 1$

8. The domain of $f(x) = \sqrt{2x+3}$ is

   (a) all the real numbers (b) all the positive real numbers (c) $(-3, \infty)$ (d) $[-\frac{3}{2}, \infty)$

Solutions

1(a) 2(c) 3(b) (we have $f(-2) = f(2) = -3$. we found two different inputs that give the same output, hence f is not injective) 4(b) (the real number -1 is not the ouput of any number, since $f(x) \geq 0$ always, no matter what real number x is) 5(d) 6(a) 7(b) 8(d)