1. The setup

Let $G$ be a connected, simply connected, semisimple linear algebraic group over the complex numbers and let $B \subseteq G$ a Borel subgroup. The flag variety is the quotient $B = G/B$. Recall that any two Borel subgroups in $G$ are conjugate and that every Borel subgroup is self-normalizing. These two facts imply that the flag variety can be understood as the set of Borel subgroups of $G$. Indeed, this follows by looking at the map

$$B \rightarrow \{ B' \subseteq G : B' \text{ is a Borel subgroup} \}, \ gB \mapsto gBg^{-1}.$$ 

In particular, the flag variety does not depend on the choice of the Borel subgroup. We may also understand the flag variety in terms of parametrizing Borel subalgebras of Lie($G$) = $\mathfrak{g}$ rather than Borel subgroups of $G$. The flag variety $B/G/B$ has a natural structure of a smooth projective variety.

**Example 1.** When $G = \text{SL}_n(\mathbb{C})$ and $B$ is the subgroup of upper-triangular matrices, then we can understand $\text{SL}_n(\mathbb{C})/B$ as the set of complete flags

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n$$

where $\dim \mathbb{C} V_i = i$. Indeed, we have an obvious action of $\text{SL}_n(\mathbb{C})$ on the set of flags which is transitive and the stabilizer of the canonical flag is the set of upper-triangular matrices. In particular, when $G = \text{SL}_2(\mathbb{C})$ we see that $B = \mathbb{P}^1$ since it parametrizes lines in $\mathbb{C}^2$.

Even though we may have many different choices of Borel subalgebras $b \subseteq \mathfrak{g}$, the quotient $b/[b,b]$ is independent of the choice of $b$. That is, given two Borel subalgebras $b_1,b_2$ there exists a canonical isomorphism given by conjugation

$$b_1/[b_1,b_1] \sim b_2/[b_2,b_2]$$

and so we shall call this quotient $\mathfrak{h} = b/[b,b]$ the abstract Cartan subalgebra of $\mathfrak{g}$. We identify any particular Cartan subalgebra $\mathfrak{h} \subseteq b \subseteq \mathfrak{g}$ with $\mathfrak{h}$ via the isomorphism

$$\mathfrak{h} \hookrightarrow b \rightarrow b/[b,b] = \mathfrak{h}.$$ 

If we fix $T \subseteq B \subseteq G$ then we have a root system associated to Lie($T$) = $\mathfrak{h}$ together with a choice of positive roots corresponding to the Borel, and we will denote by $\rho$ the half-sum of positive roots. We say that $\lambda \in \mathfrak{h}^*$ is an integral weight if

$$\langle \lambda, \alpha \rangle := 2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$$

for every root $\alpha$, where $\langle \cdot, \cdot \rangle$ is the Killing form. We say that an integral weight is dominant if $\langle \lambda, \alpha \rangle \geq 0$ for every positive root $\alpha$ and we say that $\lambda$ is regular if the previous inequality is strict. We can introduce an ordering on the set of integral weights by $\lambda \leq \mu$ if $\mu - \lambda$ is dominant.

The Weyl group $W = N_G(T)/T$ is isomorphic to the Weyl group of the root system associated to Lie($T$) = $\mathfrak{h}$ and thus does not depend on the choice of $T$. The Bruhat decomposition of $G$ is given by

$$G = \bigsqcup_{w \in W} BwB.$$
This double coset decomposition provides a stratification of the flag variety since we can cover $G/B$ by the sets $B_w = BwB/B$. These sets $B_w$ are locally closed submanifolds of the flag variety and moreover are isomorphic to the affine space $A^\ell(w)$ where $\ell(w)$ is the length of $w$ in the Weyl group. We call $B_w$ a Schubert cell and its closure $\overline{B_w}$ a Schubert variety.

Given a representation $V$ of $B$ we can form a vector bundle $G \times B V$ on the flag variety. To do this, consider the locally free $B$-action on the trivial vector bundle $G \times V$ given by $(g, v) \cdot b = (gb, b^{-1}v)$ and define $G \times B V := (G \times V)/B$ as the quotient by this action. One may readily verify that $G \times B V$ is a $G$-equivariant vector bundle. This construction yields a one-to-one correspondence

$$\{ \text{finite dimensional } B\text{-modules} \} \stackrel{\sim}{\longrightarrow} \{ G\text{-equivariant vector bundles on } B \} \quad V \mapsto G \times B V$$

In particular, $G$-equivariant line bundles are in correspondence with one-dimensional $B$-modules. But the action of the unipotent radical $N \subseteq B$ on a one-dimensional $B$-module must be trivial and thus one-dimensional $B$-modules are the same as one-dimensional $B/N \simeq T$-modules, that is, characters on the maximal torus. Given a character $\lambda : T \to \mathbb{C}$ we will denote by $L(\lambda)$ the sheaf of sections of the line bundle $G \times B C_\lambda$.

A finite dimensional representation $V$ of the Lie algebra $\mathfrak{g}$ admits a simultaneous diagonalization of the operators in $\mathfrak{h}$ and so we get a weight decomposition

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \quad V_\lambda := \{ v \in V : hv = \lambda(h)v \}.$$ 

The importance of dominant weights is that, up to isomorphism, every dominant weight $\lambda$ corresponds to a unique finite-dimensional irreducible representation $V(\lambda)$ with highest weight $\lambda$.

An element $\lambda \in \mathfrak{h}^*$ is the differential of a character in the maximal torus if and only if $\lambda$ is an integral weight. The vector bundles we have constructed will give us a geometric understanding of certain aspects of representation theory. For instance, the global sections $\Gamma(B, G \times B V)$ can be identified with the induction $\text{Ind}_B^G(V)$.

**Example 2.** In the case of $G = \text{SL}_2(\mathbb{C})$ then we know that $B = \mathbb{P}^1$ and the weight lattice is $\mathbb{Z} \rho$. We can cover $\mathbb{P}^1$ by $\{ U_0, U_1 \}$ where $U_i = \{(z_0 : z_1) : z_i \neq 0 \}$ and with choices of coordinates $z$ and $w = 1/z$ respectively. An explicit description of $L(np)$ for any $n \in \mathbb{Z}$ can be obtained as follows. Glueing the two trivial vector bundles $U_0 \times C$ and $U_1 \times C$ by $(z, u) \sim (w, v)$ if and only if $z = 1/w$ and $u = \omega^w v$, we see that $L(np) = O_{\mathbb{P}^1}(-n)$. In particular, we see that $L(np)$ is generated by global sections if $n \leq 0$ and ample if $n < 0$.

The line bundles $L(\lambda)$ associated to dominant weights will, not only enjoy good geometric properties, but will also give a geometric construction for the irreducible representation of highest weight $\lambda$. This is the content of the seminal theorem of Borel, Weil and Bott.

**Theorem 3** (Borel, Weil, Bott, [Bot57]). Let $\lambda$ be an integral weight. If $\lambda + \rho = w \cdot (\mu + \rho)$ for some $w \in W$ and some dominant weight $\mu$, then

$$H^i(\mathbb{B}, L(\lambda)) \simeq V(\mu)^*$$

as representations of $\mathfrak{g}$ and

$$H^i(\mathbb{B}, L(\lambda)) = 0$$

for all $i \neq \ell(w)$.

In particular, when $\lambda$ is a dominant weight we see that $\Gamma(B, L(\lambda)) \simeq V(\lambda)^*$ and $H^i(B, L(\lambda)) = 0$ for $i \neq 0$. 
2. SYMPLECTIC GEOMETRY

A symplectic manifold $(M, \omega)$ is a pair consisting of a smooth manifold $M$ together with a closed non-degenerate differential 2-form. The most prominent example in this class of manifolds is the cotangent space $T^*M$ of a manifold $M$: it carries a canonical 1-form $\lambda$ such that for each point $x \in M$ and $\xi \in T^*_x M$,

$$\lambda(x, \xi) = \left\langle \xi, d\pi(x, \xi) \right\rangle$$

where $\pi : T^*M \to M$ is the canonical projection. The form $\omega = d\lambda$ is a symplectic form on $T^*M$. In local coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ of the cotangent space, the symplectic form is given by

$$\omega = dx_1 \wedge d\xi_1 + \ldots + dx_n \wedge d\xi_n.$$ 

The local geometry of symplectic manifolds is well-understood, as any symplectic manifold is locally isomorphic, preserving the symplectic form, to the cotangent bundle of an affine space.

The following family of symplectic manifolds arise in the context of Lie theory, and will be very important to us.

**Example 4.** Let $G$ be a Lie group and $g$ be its Lie algebra. Let $\text{Ad} : G \to \text{Aut}(g)$ denote the adjoint representation of $G$. The coadjoint representation $K : G \to \text{Aut}(g^*)$ is defined as

$$\left\langle K(g)\lambda, \xi \right\rangle = \left\langle \lambda, \text{Ad}(g^{-1})\xi \right\rangle$$

for $g \in G$, $\lambda \in g^*$ and $\xi \in g$. Any orbit $\Omega \subseteq g^*$ for the coadjoint action is an immersed submanifold which comes equipped with a canonical symplectic form (sometimes called the Kirillov-Kostant-Souriau symplectic structure). For each point $\alpha \in \Omega$, we may identify $\Omega$ with $G/G^\alpha$ where $G^\alpha$ is the isotropy group of $\alpha$ with Lie($G^\alpha$) = $\mathfrak{g}^\alpha$ and so $T_\alpha \Omega = g/\mathfrak{g}^\alpha$. We can define a skew-symmetric form

$$\omega_\alpha : g \times g \to \mathbb{C}, \quad \omega_\alpha(x, y) = \alpha([x, y])$$

which descends to $g/\mathfrak{g}^\alpha$ and thus $\alpha \mapsto \omega_\alpha$ gives us a non-degenerate 2-form on $\Omega$. Computing $d\omega = 0$ is straightforward applying Cartan’s formula for the exterior differential and Jacobi’s identity.

Observe that $g^*$ comes equipped with a canonical Poisson bracket given by the Lie bracket on the space of linear functions $g$ and the coadjoint orbits are the symplectic leaves of this Poisson structure.

Let $(M, \omega)$ be a symplectic manifold. For a regular function $f \in \mathcal{O}(M)$ we can associate a Hamiltonian vector field $\xi_f$ that satisfies $\omega(\xi_f, \cdot) = -df$. Thus the symplectic structure induces a Poisson bracket on the ring of functions $\mathcal{O}(M)$ via

$$\{f, g\} = \omega(\xi_f, \xi_g).$$

We say that an action of a Lie group $G$ on $M$ is symplectic if it preserves the symplectic form, that is $\omega(\xi, y) = \omega(g\xi, gy)$ for all $\xi, y \in T_m M$, $m \in M$, $g \in G$. This induces an infinitesimal action of the Lie algebra $g \to \text{Vect}(M)$. We say that the $G$-action is Hamiltonian if there exists a Lie algebra morphism $H : g \to \mathcal{O}(M)$, $x \mapsto H_x$ such that the following diagram of Lie algebra maps commute:

$$\begin{array}{ccc}
\mathcal{O}(M) & \xrightarrow{H} & \text{Vect}(M) \\
\mathcal{O}(M) & \xleftarrow{\text{Ad}} & \text{Alg}(g)
\end{array}$$

The moment map is defined as

$$\mu : M \to g^*, \ m \mapsto (x \in g \mapsto H_x(m)).$$

If $G$ is connected, the moment map is $G$-equivariant with respect to the coadjoint action on $g^*$. 
Example 5. In the classical setting $G = \text{SO}_3(\mathbb{R})$, the cotangent bundle $T^*\mathbb{R}^3$ possess a symplectic structure and the obvious action of $\text{SO}_3(\mathbb{R})$ on $T^*\mathbb{R}^3$ is Hamiltonian. The moment map is

$$\mu: T^*\mathbb{R}^3 \to \mathfrak{so}_3^*, \ \mu(q,p) = q \times p$$

namely the usual angular momentum once we identify $(\mathfrak{so}_3, [\cdot,\cdot])$ with $(\mathbb{R}^3, \times)$. More generally, when a Lie group acts on any manifold $X$ it induces a symplectic action on $M = T^*X$ which is moreover Hamiltonian. Indeed, if $\nu: \mathfrak{g} \to \text{Vect}(T^*X)$ is the infinitesimal action and $\lambda$ is the canonical 1-form on $T^*X$ then the Hamiltonian is given by $x \mapsto H_x = \lambda(\nu_x)$. In particular, we obtain a moment map $\mu: T^*X \to \mathfrak{g}^*.$

In the case of $P \subseteq G$ a Lie subgroup and $G$ acting on $T^*(G/P)$ we can give a more explicit description of the moment map. Let $p^\perp$ be the annihilator of $p = \text{Lie}(P)$ in $\mathfrak{g}^*.$

**Proposition 6.** There is a natural $G$-equivariant isomorphism

$$T^*(G/P) \xrightarrow{\sim} G \times^P p^\perp,$$

where $P$ acts on $p^\perp$ by the coadjoint action. Under the above isomorphism, the moment map $\mu$ is given explicitly by

$$(g, \alpha) \mapsto g \alpha g^{-1}, \ g, \alpha \in p^\perp.$$

3. The Springer resolution

Given a semisimple algebraic group $G$ with Lie algebra $\mathfrak{g}$, let us fix a Borel subgroup $B \subseteq G$ with its Lie algebra $\mathfrak{b} = \text{Lie}(B)$ as a base-point of the flag variety $B = G/B$. Since the flag variety parametrizes Borel subalgebras of $\mathfrak{g}$, we can consider the incidence variety given by

$$\widetilde{\mathfrak{g}} = \{ (x, b') \in \mathfrak{g} \times B : x \in b' \},$$

together with the two projection maps $\mu: \widetilde{\mathfrak{g}} \to \mathfrak{g}$ and $\pi: \widetilde{\mathfrak{g}} \to B$.

The projection map $\pi: \widetilde{\mathfrak{g}} \to B$ makes it a $G$-equivariant vector bundle with fiber $\mathfrak{b}$, and the assignment $(g, x) \mapsto (g x g^{-1}, gB/B)$ gives a $G$-equivariant isomorphism $G \times^B \mathfrak{b} \xrightarrow{\sim} \widetilde{\mathfrak{g}}$.

On the other hand, the projection map $\mu: \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is a proper map with fibers $\mu^{-1}(x) = B_x$ the set of Borel subalgebras that contain $x$. The dimensions of the fibers may vary.

Recall that an element $x \in \mathfrak{g}$ is regular if the dimension of the centralizer $\dim Z_{\mathfrak{g}}(x)$ is equal to the rank of $\mathfrak{g}$ (that is, the dimension of any Cartan subalgebra). We denote by $\mathfrak{g}^{\text{reg}}$ the set of regular elements in $\mathfrak{g}$. We say that $x \in \mathfrak{g}$ is semisimple if $\text{ad}(x)$ is diagonalizable and we say $x$ is nilpotent if $\text{ad}(x)$ is nilpotent.

**Proposition 7.** The set $\mathfrak{g}^{\text{reg}}$ of regular semisimple elements is a $G$-stable Zariski open subset.

**Proof.** Given $x \in \mathfrak{g}$ consider the characteristic polynomial $\det(t - \text{ad}(x))$ of the operator $\text{ad}(x)$. Since $Z_{\mathfrak{g}}(x) = \ker(\text{ad}(x))$ it follows that $\dim \ker(\text{ad}(x)) \geq \text{rk} \mathfrak{g} = r$ and so we can expand

$$\det(t - \text{ad}(x)) = t^r P_r(x) + t^{r+1} P_{r+1}(x) + \cdots$$

where $P_i(x)$ are $G$-invariant polynomials on $\mathfrak{g}$. We claim that $\mathfrak{g}^{\text{reg}}$ is precisely the complement of $V(P_r)$, which implies that it is a $G$-invariant Zariski open subset. Indeed, if we consider the Jordan decomposition $x = s + n$ with $s$ semisimple, $n$ nilpotent and $[s,n] = 0$ then the characteristic polynomial of $\text{ad}(x)$ and $\text{ad}(s)$ coincide. If $s$ is regular then $Z_{\mathfrak{g}}(s)$ is a Cartan subalgebra and so $n \in Z_{\mathfrak{g}}(s)$ must vanish, from which $x = s$ is semisimple and regular. On the other hand, if $s$ is not regular then the characteristic polynomial of $\text{ad}(s)$ will have a zero of higher order than $r$.

**Example 8.** In the case $\mathfrak{g} = \mathfrak{sl}_n$ the set of regular elements are those whose Jordan blocks have different eigenvalues. In particular, regular semisimple elements in $\mathfrak{sl}_n$ are traceless matrices with $n$ different eigenvalues.

We can illustrate the fibers of the projection $\mu$ in this case. Recall that in the $\mathfrak{sl}_n$ case, the flag variety $B$ parametrizes complete flags $F = (0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n)$ and the corresponding
Borel subalgebra is the stabilizer in \( \mathfrak{g} \) of this flag, that is, \( xF_i \subseteq F_i \) for each \( i \). Therefore, the incidence variety consists of pairs of elements \((x,F)\) with \( x \in \mathfrak{sl}_n \) and \( F \) a flag stabilized by \( x \). We see that the fiber \( \mu^{-1}(x) \) is the set of flags stabilized by \( x \). In the case that \( x \) is a regular semisimple element we can take the eigenspaces \( V_1, \ldots, V_n \) so given a permutation \( \{i_1, \ldots, i_n\} \) of \( \{1, \ldots, n\} \) we obtain a flag

\[
F_x = \bigoplus_{j=1}^{\ell} V_{i_j}
\]

and moreover every flag stabilized by \( x \) must be of this form. Hence the fiber \( B_x \) consists of \( n! \) points for any regular semisimple element \( x \). In fact, when restricted to the open set of regular semisimple elements, \( \mu \) becomes a covering map with Galois group \( S_n \). This illustrates the fact that the dimensions of the fibers of \( \mu \) may vary drastically, since \( \mu^{-1}(0) \) is the whole flag variety whereas the fibers of regular semisimple elements are finite.

The behaviour of the fibers exhibited by the previous example holds in greater generality. If we denote by \( \tilde{\mathfrak{g}}^{rs} = \mu^{-1}(\mathfrak{g}^{rs}) \) we can restrict our attention to the map \( \mu : \tilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs} \) a principal \( W \)-bundle.

**Proposition 9.** For each \( x \in \mathfrak{g}^{rs} \), there is a canonical free \( W \)-action on \( \mu^{-1}(x) \) making the projection \( \mu : \tilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs} \) a principal \( W \)-bundle.

**Proof.** If \( x \in \mathfrak{g}^{rs} \) then its centralizer is a Cartan subalgebra \( \mathfrak{h} \) and we can consider the associated root system. An element in the fiber \( \mu^{-1}(x) \) is simply a choice of a Borel subalgebra that contains \( x \), which is equivalent to a choice a Borel subalgebra containing \( \mathfrak{h} \). The choice of a Borel subalgebra containing a fixed Cartan subalgebra amounts to the choice of a base in the root system, and we know that the Weyl group acts simply transitively on bases. This gives the desired free canonical action.

The nilpotent cone \( \mathcal{N} \subseteq \mathfrak{g} \) is the set of nilpotent elements in \( \mathfrak{g} \). It is a closed \( G \)-stable subvariety of \( \mathfrak{g} \) and is also stable with respect to the \( \mathbb{C}^* \) action given by dilations. This variety is always singular at the origin. Let us denote

\[
\tilde{\mathcal{N}} = \mu^{-1}(\mathcal{N}) = \{(x,b') \in \mathcal{N} \times \mathcal{B} : x \in b' \}.
\]

By fixing a Borel subalgebra \( \mathfrak{b} \), the fiber under the second projection \( \pi : \tilde{\mathcal{N}} \to \mathcal{B} \) consists of all the nilpotent elements in \( \mathfrak{b} \), which is precisely the nilradical \( n = [\mathfrak{b}, \mathfrak{b}] \). This means that \( \tilde{\mathcal{N}} \) is a vector bundle with fiber \( n \) over \( \mathcal{B} \). Since any nilpotent element in \( \mathfrak{g} \) is \( G \)-conjugate into \( n \), it follows that \( \tilde{\mathcal{N}} \) is \( G \)-equivariantly isomorphic to \( G \times B \). Moreover, if we identify \( \mathfrak{g} \simeq \mathfrak{g}^* \) via a \( G \)-equivariant isomorphism (for instance, via the pairing given by the Killing form), we can identify \( n \simeq \mathfrak{b}^\perp \) and so the description of the cotangent bundle of the flag variety (see Proposition 6) yields a \( G \)-equivariant isomorphism

\[
\tilde{\mathcal{N}} \simeq G \times B \n \simeq G \times B \mathfrak{b}^\perp \simeq T^\ast \mathcal{B}
\]

and the description of the moment map shows that, under this isomorphism, the projection \( \mu : \tilde{\mathcal{N}} \to \mathcal{N} \) is the moment map. It is also clear that \( \mu : \tilde{\mathcal{N}} \to \mathcal{N} \) is surjective since any nilpotent element belongs to some Borel subalgebra. The map \( \mu : T^\ast \mathcal{B} = \tilde{\mathcal{N}} \to \mathcal{N} \) is called the Springer's map.

**Example 10.** In the case of \( G = \text{SL}_2(\mathbb{C}) \), the nilpotent cone consists of traceless matrices with vanishing determinant

\[
\mathcal{N} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a^2 + bc = 0 \right\}
\]

which is the familiar equation of a cone (perhaps more familiar after the change of coordinates \( b = x + iy, c = x - iy \)). In this case, we can also describe \( \tilde{\mathcal{N}} \) quite explicitly. A flag in \( \mathbb{C}^2 \) consists of simply a line. Thus the fiber \( \pi^{-1}(L) \) is precisely the set of nilpotent operators that preserve the line
L. If we fix a generator \( v \) of the line \( L \) and complete it to a basis \( \{ v, w \} \) of \( \mathbb{C}^2 \), a nilpotent operator \( x \) preserves \( L \) if and only if (changing the basis to \( \{ v, w \} \)) it is of the form

\[
[x]_{\{v,w\}} = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}, \quad \lambda \in \mathbb{C}
\]

from which we see that the fiber \( \pi^{-1}(L) \) is a line through the origin in the nilpotent cone \( \mathcal{N} \). On the other hand, if we fix a point \( x \in \mathcal{N} \), the fiber \( \mu^{-1}(x) \) consists of lines that are preserved by \( x \). Since \( x \) is a nilpotent operator, if \( x \neq 0 \) then the only line that can be preserved by \( x \) is \( L = \ker x \), whereas if \( x = 0 \) every line is preserved. Since every line \( L \) gets associated to a line in \( \mathcal{N} \) via \( \pi \), this discussion shows that the map \( \mu : \mathcal{N} \to \mathcal{N} \) can be identified with the blow-up of the cone \( \mathcal{N} \) at the origin. In particular, the map \( \mu : \mathcal{N} \to \mathcal{N} \) is a resolution of singularities.

Thus far, we have the following picture

\[
\begin{array}{ccc}
T^*B = \mathcal{N} & \hookrightarrow & \bar{\mathfrak{g}} \\
\mu \downarrow & & \downarrow \mu \\
\mathcal{N} & \to & \mathfrak{g} [\mathfrak{g}^r, \mathfrak{g}^r]
\end{array}
\]

where the map on the right is a covering map with Galois group \( W \) and the map on the left is the moment map.

To understand the geometry of the nilpotent cone, we shall introduce the Chevalley restriction map, following [CG10]. If we fix a maximal torus \( T \subseteq G \) with its corresponding Cartan subalgebra \( \text{Lie}(T) = \mathfrak{h} \), then the restriction map \( C[\mathfrak{g}] \to C[\mathfrak{h}] \) induces a morphism \( C[\mathfrak{g}]^G \to C[\mathfrak{h}]^W \) where \( W = N_G(T)/T \) is the Weyl group.

**Theorem 11** (Chevalley restriction theorem). For any Cartan subalgebra \( \mathfrak{h} \subseteq \mathfrak{g} \) the restriction map yields a canonical graded algebra isomorphism

\[
C[\mathfrak{g}]^G \sim \to C[\mathfrak{h}]^W.
\]

**Proof.** First, let us show that the restriction map is injective. Take \( P \in C[\mathfrak{g}]^G \) such that \( P|_{\mathfrak{h}} = 0 \). Since every semisimple regular element is \( G \)-conjugate to some element in \( \mathfrak{h} \), it follows that \( P \) vanishes on \( \mathfrak{g}^r \), but we know that the set of regular semisimple elements is open and dense, from which the injectivity follows.

Now, let us see the surjectivity. Take some \( W \)-invariant polynomial \( P \) on \( \mathfrak{h} \), which we may identify as some \( W \)-invariant polynomial \( P \) on the abstract Cartan \( \mathfrak{h} \). The advantage of passing to the abstract Cartan is that we have a map \( \nu : \bar{\mathfrak{g}} \to \mathfrak{h} \) given by

\[
(x, b) \in \bar{\mathfrak{g}} \mapsto \mathfrak{h} \in b/[b, b] = \mathfrak{h}
\]

and so we can lift \( P \) to a polynomial \( \nu^*P \) in \( \bar{\mathfrak{g}} \). We claim that \( \nu^*P \) is \( G \)-invariant. Indeed, this is clear once we identify \( \nu \) with the projection \( G \times B b \to b/[b, b], (g, x) \mapsto \mathfrak{h} \) via the \( G \)-equivariant isomorphism \( \bar{\mathfrak{g}} \simeq G \times B b \). We would like to descend the polynomial \( \nu^*P \) to \( \mathfrak{g} \) in order to obtain a polynomial \( Q \) on \( \mathfrak{g} \) such that \( Q|_{\mathfrak{h}} = P \). We shall do a slight variation of this. If we restrict \( \nu^*P \) to \( \mathfrak{g}^r \) we obtain a \( W \)-invariant polynomial since \( \nu \) commutes with the \( W \)-action and \( P \) is \( W \)-invariant in \( \mathfrak{h} \). Then \( \nu^*P|_{\mathfrak{g}^r} \) descends to a rational function \( Q \) on \( \mathfrak{g}^r \) since the map \( \mu : \mathfrak{g}^r \to \mathfrak{g}^r \) is a cover with Galois group \( W \) and so \( C(\mathfrak{g}^r)^W = C(\mathfrak{g}^r) \). Notice that \( Q \) is \( G \)-invariant because \( \mu \) respects the \( G \)-actions. We claim that \( Q \) extends to a \( G \)-invariant polynomial on \( \mathfrak{g} \). Indeed, we only need to show that for any relatively compact \( K \subseteq \mathfrak{g} \) the restriction of \( Q \) to \( K \cap \mathfrak{g}^r \) is bounded and hence it has no poles. But this follows from the properness of \( \mu \) and the fact that \( \mu^*Q = \nu^*P|_{\mathfrak{g}^r} \) is a polynomial. Therefore we constructed a \( G \)-invariant polynomial \( Q \) on \( \mathfrak{g} \) and we may easily verify that the restriction to \( \mathfrak{h} \) is the original polynomial \( P \), as desired. \( \blacksquare \)
We denote by \( \mathfrak{N} \parallel W \) the variety \( \text{Spec}(\mathbb{C}[\mathfrak{n}]^W) \). The classical Chevalley-Shepherd-Todd theorem [Bou02, Ch. 5] tells us that the algebra of invariants \( \mathbb{C}[\mathfrak{n}]^W \) is a polynomial algebra on \( \text{rk}(\mathfrak{g}) \) variables and \( \mathbb{C}[\mathfrak{n}] \) is free over \( \mathbb{C}[\mathfrak{n}]^W \). If we reverse the isomorphism in Theorem 11, we obtain an injective map \( \mathbb{C}[\mathfrak{n}]^W \rightarrow \mathbb{C}[\mathfrak{g}] \) which induces a map \( \rho : \mathfrak{g} \rightarrow \mathfrak{N} \parallel W \) on the geometric side. Notice that we essentially constructed the map \( \rho \) in the proof of Theorem 11, by setting \( \rho^*(P) \) to be the function \( \nu^*(P) \) described via \( \mu \) to \( \nu^* \) and extended to the whole \( \mathfrak{g} \). All in all, we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\nu} & \mathfrak{N} \\
\rho \downarrow & & \downarrow \omega \\
\mathfrak{N} \parallel W & \xrightarrow{\mu} & \mathfrak{g}
\end{array}
\]

**Example 12.** In the case of \( \mathfrak{g} = \mathfrak{sl}_n \), the Weyl group is \( W = S_n \) and we may identify the Cartan subalgebra consisting of traceless diagonal matrices with the hyperplane inside of \( \mathbb{C}^n \) whose coordinates add to 0. Hence \( \mathfrak{N} \parallel W \) becomes identified with \( \mathbb{C}^{n-1} \parallel S_n \). We can see explicitly that this quotient

\[
\mathbb{C}^{n-1} \parallel S_n = \text{Spec}(\mathbb{C}[x_1, \ldots, x_n]/(x_1 + \ldots + x_n = 0))^{S_n}
\]

is an affine space of dimension \( \text{rk}(\mathfrak{sl}_n) = n - 1 \), since the ring of \( S_n \)-invariants in \( \mathbb{C}[x_1, \ldots, x_n] \) is spanned by the elementary symmetric polynomials. If we associate for any \( x \in \mathfrak{sl}_n \) the unordered set of eigenvalues \( \{x_1, \ldots, x_n\} \), then \( x_1 + \ldots + x_n = 0 \) and we get the map \( \rho : \mathfrak{sl}_n \rightarrow \mathbb{C}^{n-1} \parallel S_n \), or we may equivalently describe it as the map that associates a matrix to its characteristic polynomial. It is clear in this case that the fiber \( \rho^{-1}(0) \) consists of the set of nilpotent matrices in \( \mathfrak{sl}_n \).

Notice that the fiber \( \rho^{-1}(0) \) is precisely the nilpotent cone as a set. Indeed, if \( x \in \mathfrak{N} \) then choosing some lift \( \tilde{x} = (x, b) \in \tilde{\mathfrak{g}} \) we have that \( x \in [b, b] \) since \( x \in b \) is nilpotent and so \( \nu(\tilde{x}) = 0 \). The commutativity of the previous diagram implies that \( \rho(x) = \omega(\nu(\tilde{x})) = 0 \). On the other hand, if \( x \in \rho^{-1}(0) \), consider once again some lift \( \tilde{x} = (x, b) \). Since \( \omega^{-1}(0) = \{0\} \), the fact that \( \rho(x) = \omega(\nu(\tilde{x})) = 0 \) implies that \( \nu(\tilde{x}) = 0 \) from which \( x \in [b, b] \) is nilpotent as desired.

A point \( x \in \mathfrak{N} \parallel W \) corresponds to a morphism \( \chi : \mathbb{C}[\mathfrak{n}]^W \rightarrow \mathbb{C} \), or equivalently, to a morphism \( \chi : \mathbb{C}[\mathfrak{g}]^G \rightarrow \mathbb{C} \) and the maximal ideal of the point corresponds to \( \ker \chi \). In the case of \( 0 \in \mathfrak{N} \parallel W \), the corresponding morphism is the evaluation at 0 and hence the maximal ideal of 0 in \( \mathfrak{N} \parallel W \) is the set \( \mathbb{C}[\mathfrak{g}]_+^G \) of \( G \)-invariant polynomials on \( \mathfrak{g} \) without constant term. The geometric observation that \( \mathfrak{N} = \rho^{-1}(0) \) yields the following coordinate-free description of the nilpotent elements in \( \mathfrak{g} \), due to Kostant.

**Theorem 13** (Kostant). An element \( x \in \mathfrak{g} \) is nilpotent if and only if for every \( P \in \mathbb{C}[\mathfrak{g}]_+^G \) we have \( P(x) = 0 \).

**Proof.** This is immediate since structure sheaf on \( \rho^{-1}(0) \) is given by the ring \( \mathbb{C}[\mathfrak{g}]/\mathbb{C}[\mathfrak{g}]_+^G \cdot \mathbb{C}[\mathfrak{g}] \) and the fiber \( \rho^{-1}(0) \) is set-theoretically the nilpotent cone.

The nilpotent cone is an irreducible variety since \( T^*B \) is a smooth, connected (hence irreducible) variety and \( \mathfrak{N} \) is its image under \( \mu \). Moreover, it implies that \( \dim \mathfrak{N} \leq \dim T^*B = 2 \dim \mathfrak{n} \). On the other hand, since \( \mathfrak{N} \) is the zero fiber under \( \rho \), by Chevalley’s theorem on the dimension of the fibers [Gro66, Lemme 13.1.1], it follows that \( \dim \mathfrak{N} \geq \dim \mathfrak{g} - \dim \mathfrak{N} \parallel W = 2 \dim \mathfrak{n} \). Hence \( \mathfrak{N} \) is an irreducible variety of dimension \( 2 \dim \mathfrak{n} \) and since it is cut out by the rkq equations of \( \mathbb{C}[\mathfrak{g}]_+^G \) it follows that it is a complete intersection and thus Cohen-Macaulay.

**Proposition 14.** The number of \( G \)-orbits in \( \mathfrak{N} \) is finite and the set of regular nilpotent elements \( \mathfrak{N}^{\text{reg}} \) forms a single open dense \( G \)-orbit in \( \mathfrak{N} \).
Proof. In the case of $G = \text{GL}_n$, the Lie algebra $\mathfrak{gl}_n$ has finitely many nilpotent conjugacy classes by looking at their Jordan normal forms. If we embed our Lie algebra $\mathfrak{g}$ into $\mathfrak{gl}_n$ for some $n$, then for any $\text{GL}_n$-orbit $O$ in $\mathfrak{gl}_n$, the irreducible components of the intersection $O \cap \mathfrak{g}$ are $G$-orbits, and hence $\mathcal{N}$ is a union of finitely many $G$-orbits. For more details, see [Dix96, Lemma 8.1.2].

Now, since $\mathcal{N}$ is irreducible and consists of finitely many nilpotent $G$-orbits, it follows that there exists a unique open dense orbit $O$. If $x \in O$ then we see that
\[
2 \dim n = \dim \mathcal{N} = \dim O = \dim G - \dim Z_G(x)
\]
and so $\dim Z_G(x) = \text{rk}(\mathfrak{g})$ from which $x$ must be regular. $
$

Before we prove our main theorem we will need to establish the following lemma.

Lemma 15. Let $e_1, \ldots, e_{\ell} \in n$, where $\ell = \text{rk}(\mathfrak{g})$, be root vectors corresponding to positive simple roots with respect to $b$. Then there exists a regular element $e \in n$ such that its image $\overline{e}$ in $n/[n,n]$ is of the form $\lambda_1 \overline{e}_1 + \ldots + \lambda_{\ell} \overline{e}_\ell$ with $\lambda_i \neq 0$ for all $i$.

Proof. Both the subset of regular elements in $n$ and the subset $\{ x \in n : \overline{x} = \sum \lambda_i \overline{e}_i, \lambda_i \in \mathbb{C}^\times \}$ are Zariski open and so they have non-trivial intersection. $
$

Remark 16. In fact, it is known [Kos59, Theorem 5.3] that the set of regular elements in $n$ is exactly the set $\{ x \in n : \overline{x} = \sum \lambda_i \overline{e}_i, \lambda_i \in \mathbb{C}^\times \}$.

We are now in conditions of proving the following theorem.

Theorem 17. The map $\mu : T^* B \to \mathcal{N}$ is a resolution of singularities.

Proof. Since $T^* B$ is a smooth variety and $\mu$ is a proper map, it will be enough to prove that $\mu$ is an isomorphism over $\mathcal{N}^{\text{reg}}$. In more concrete terms, this means that each regular nilpotent element is contained in a unique Borel subalgebra.

Notice that $\mu : T^* B \to \mathcal{N}$ is a surjective map between irreducible varieties of the same dimension, from which the generic fiber is zero-dimensional, that is, $B_x = \mu^{-1}(x)$ is a discrete set for generic $x \in \mathcal{N}$. Moreover, the fibers of conjugate elements in the nilpotent cone are canonically isomorphic, and hence it will suffice to prove that for a specific regular nilpotent element in $\mathcal{N}$ the fiber consists of one point. Let $e_1, \ldots, e_{\ell} \in n$ be root vectors corresponding to positive simple roots with respect to $b$ and pick a regular element $e \in e$ as in the previous lemma. If we take $h \in \mathfrak{h}$ such that $[h,e] = e$ and $h$ is regular, then the one-parameter subgroup corresponding to $Ch \subseteq \mathfrak{g}$ stabilizes the fiber $\mu^{-1}(e)$. Since the fiber is discrete every point must be fixed by this one-parameter subgroup. This means that $h \in b'$ for all $b' \in \mu^{-1}(e)$ which in turn implies that $\mathfrak{h} \subseteq b'$ for $h$ is regular. Hence $\mu^{-1}(e) \subseteq W \cdot b$. If we look at the associated root system to our fixed Cartan subalgebra $\mathfrak{h}$, we know that the set $W \cdot b$ consists of the different possible bases and hence to a choice of signs of the root vectors $e_i$. In particular, the element $e$ will never be positive in any other base which is not the corresponding to $b$, or equivalently, the only Borel subalgebra containing $e$ is $b$. This concludes the proof. $
$

We claim that the nilpotent cone $\mathcal{N}$ is normal. By Serre’s criterion [Gro65, Théorème 5.8.6], since $\mathcal{N}$ is Cohen-Macaulay, it suffices to prove that it is regular in codimension 1. That is, we need to show that $\text{codim}(\mathcal{N} \setminus \mathcal{N}^{\text{reg}}) \geq 2$. But $\mathcal{N}$ consists of finitely many $G$-orbits and coadjoint orbits are symplectic with respect to the natural Poisson structure on $\mathfrak{g}$, and thus they must be even dimensional from which the claim follows.

Lemma 18. Let $f : X \to Y$ be a birational projective morphism of noetherian integral schemes and assume that $Y$ is normal. Then $f_* \mathcal{O}_X = \mathcal{O}_Y$.

Proof. The question is local in $Y$ so we may assume that $Y$ is affine, $Y = \text{Spec}(A)$ for some ring $A$. Then, as $f$ is proper, $f_* \mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_Y$-algebras from which $B = \Gamma(Y, f_* \mathcal{O}_X)$ is a finitely generated $A$-module. But then $A$ and $B$ are integral domains and by birationality they have the same quotient field. Since $A$ is integrally closed, we must have $A = B$ as desired. $

Corollary 19. The Springer resolution $\mu : T^*B \to N$ yields an isomorphism $\Gamma(N, \mathcal{O}_N) \simeq \Gamma(T^*B, \mathcal{O}_{T^*B})$.

Proof. Since $\mu : T^*B \to N$ is a resolution of singularities and $N$ is normal, by Lemma 18 it follows that the canonical map

$$\Gamma(N, \mathcal{O}_N) \xrightarrow{\sim} \Gamma(T^*B, \mathcal{O}_{T^*B}) = \Gamma(B, \text{Sym}\mathcal{O}_B \mathcal{T}_B)$$

is an isomorphism.\[\square\]

4. D-modules on the flag variety

Let us recall the basic definitions on differential operators on a smooth algebraic variety.

Let $X$ be a smooth algebraic variety over $\mathbb{C}$. The tangent sheaf $\mathcal{T}_X$ on $X$ defined on open affine subsets $U = \text{Spec}(A)$ by $\Gamma(U, \mathcal{T}_X) = \text{Der}_C(A)$. The sections of $\mathcal{T}_X$ over some open set $U$ are called vector fields over $U$. Notice that we have an action of vector fields $\mathcal{T}_X$ on regular functions $\mathcal{O}_X$. Indeed, this action is clear when we restrict to open affine subsets $U = \text{Spec}(A)$ then derivations act on $A$. Therefore we can embed the tangent sheaf $\mathcal{T}_X$ into $\text{End}_C(\mathcal{O}_X)$.

A differential operator on $X$ will be defined inductively in terms of its order. A differential operator of order 0 is a map $\mathcal{O}_X \to \mathcal{O}_X$ of the form $h \mapsto fh$ for some $f \in \mathcal{O}_X$ fixed (abusing of the notation we will also call this map $f$). A differential operator of order less or equal to $k$ is a $C$-linear map $L : \mathcal{O}_X \to \mathcal{O}_X$ such that the commutator $[L, f] = L \circ f - f \circ L$ is a differential operator of order less or equal to $k - 1$ for any regular function $f$. We shall denote by $\mathcal{D}(X)$ the ring of differential operators on $X$. This ring is non-commutative and has a filtration $\mathcal{F}_k \mathcal{D}(X)$ by order of operators. We define a sheaf $\mathcal{D}_X$ of differential operators on $X$ via $\Gamma(U, \mathcal{D}_X) = \mathcal{D}(U)$, Notice that this sheaf is the subsheaf of $\text{End}_C(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and $\mathcal{T}_X$ and that the order filtration of the rings yields a filtration of the sheaf.

A left $D$-module on $X$ is a quasi-coherent $\mathcal{O}_X$-module $M$ equipped with a left $\mathcal{D}_X$-action compatible with the $\mathcal{O}_X$-action. More concretely, for $m \in M$, $f \in \mathcal{O}_X$, $\xi \in \mathcal{T}_X$ we have

$$f \cdot (\xi \cdot m) = (f \cdot \xi) \cdot m \quad \text{and} \quad \xi \cdot (f \cdot m) - f \cdot (\xi \cdot m) = \xi(f) \cdot m.$$

Although the rings of differential operators are not commutative, they are not very far away from being so. We shall say that a filtered ring $A = \bigcup_k F_k A$ is almost commutative if its associated graded $\text{gr}(A) = \bigoplus_k F_k / F_{k-1}$ is a commutative ring.

Proposition 20. Given a smooth algebraic variety $X$, the associated graded $\text{gr}(\mathcal{D}_X)$ with respect to the order filtration is isomorphic to the symmetric algebra of the tangent sheaf $\text{Sym}_{\mathcal{O}_X}(\mathcal{T}_X)$.

Given an almost commutative filtered algebra $A$, the bracket satisfies $[F_k A, F_l A] \subseteq F_{k+l-1} A$ and passing down to the quotient we may see that the commutator defines a Poisson bracket on the associated graded $\text{gr}(A)$. In the situation as above, the Poisson bracket of $\text{gr}(\mathcal{D}_X)$ inherited from the order filtration coincides with the symplectic Poisson bracket on regular functions of the cotangent bundle. In this sense we say that the ring of differential operators $\mathcal{D}_X$ is a quantization of the cotangent bundle.

Suppose that we have an algebraic group $G$ acting on some variety $X$. This induces an infinitesimal action $\mathfrak{g} \to \text{Vect}(X)$ which can be extended to a map $\mathcal{U}_\mathfrak{g} \to \Gamma(X, \mathcal{D}_X)$ by the universal property of the enveloping algebra. The universal enveloping algebra comes with a canonical filtration for which the Poincaré-Birkhoff-Witt theorem tells us that the associated graded is the symmetric algebra on $\mathfrak{g}$, that is, the ring of polynomial functions on $\mathfrak{g}^*$. The action map $\Phi : \mathcal{U}_\mathfrak{g} \to \Gamma(X, \mathcal{D}_X)$ is clearly filtration preserving and thus it descends to a map $\mathbb{C}[\mathfrak{g}^*] \to \mathcal{O}(T^*X)$. Unraveling the definitions we can see that this map is precisely the moment map. Hence we can say that the action map $\Phi : \mathcal{U}_\mathfrak{g} \to \Gamma(X, \mathcal{D}_X)$ is a quantization of the moment map.

Theorem 21. Let $m \subseteq \mathcal{L}_\mathfrak{g}$ be the kernel of the character by which $\mathcal{L}_\mathfrak{g}$ acts on the trivial $\mathcal{U}_\mathfrak{g}$-module $\mathcal{C}$. Then the action map yields an isomorphism $\Phi : \mathcal{U}_\mathfrak{g} / m \mathcal{U}_\mathfrak{g} \xrightarrow{\sim} \Gamma(B, \mathcal{D}_B)$.\[\square\]
Proof. Note that \( Z_\mathfrak{g} = U\mathfrak{g}^G \) and so \( \Phi(z) \in \Gamma(B, D_B)^G \) for any \( z \in Z_\mathfrak{g} \). However, \( \Gamma(B, D_B)^G = C \) since we can descend to the associated graded where we have that \( \Gamma(N, O_N)^G = C \) because the nilpotent cone has an open dense \( G \)-orbit. By applying the theorem of Borel-Weil and Harish-Chandra isomorphism [Dix96, Proposition 7.4.4] we see that the action of the center \( Z_\mathfrak{g} \) on \( \Gamma(B, O_B) \) must be trivial and hence the map \( \Phi \) factors through \( U\mathfrak{g}/\mathfrak{m}U\mathfrak{g} \).

Now the theorem follows by noticing that \( \Phi \) is a quantization of the moment map. More precisely, we have a chain of maps

\[
\Gamma(N, O_N) = C[\mathfrak{g}]/C[\mathfrak{g}]^+ C[\mathfrak{g}] \to grU\mathfrak{g}/\mathfrak{m}U\mathfrak{g} \to gr\Gamma(B, D_B) \hookrightarrow \Gamma(B, grD_B) = \Gamma(B, Sym_{O_B} T_B)
\]

such that the composition is the morphism induced by the moment map. Applying Corollary 19 the result follows.

In fact, the global sections of \( D_B \) contain all of the information in the following sense:

We say that a variety \( X \) is \( D \)-affine if the global section functor \( \Gamma : \mathcal{F} \mapsto \Gamma(X, \mathcal{F}) \) is an equivalence of categories between the category of \( D \)-modules and that of \( \Gamma(X, \mathcal{D}) \)-modules. Notice that Serre’s criterion for affineness [Gro66, Théorème 5.2.1] tells us that \( O_X \)-affine varieties are precisely affine varieties.

**Theorem 22** (Beilinson-Bernstein localization [BB81]). The flag variety \( B \) is \( D_B \)-affine. In particular, the category \( U\mathfrak{g}_0 \)-mod of \( \mathfrak{g} \)-modules with trivial central character is equivalent to the category of \( D_B \)-modules.

**Remark 23.** The localization functor \( Loc : D_B \text{-mod} \to U\mathfrak{g}_0 \text{-mod} \) given by

\[
Loc(M) = D_B \otimes_{U\mathfrak{g}} M
\]

is the inverse functor for the global section functor in the above equivalence. It deserves the name “localization” because of the similarity with the functor \( R \text{-mod} \to QCoh(Spec(R)) \) given by \( O_{Spec(R)} \otimes_R \).

**References**


