Problem 1 Let $A \subset \mathbb{R}^n$ be non-empty and $f_n : A \rightarrow \mathbb{R}$ a sequence of functions such that

(i) There is $\alpha \in (0, 1]$ and $C > -0$ such that $|f_n(x) - f_n(y)| \leq C|x - y|^\alpha$ for any $x, y \in A$ and $n \in \mathbb{N}$.

(ii) $f_n$ converges pointwise to $f : A \rightarrow \mathbb{R}$.

Show that also $f$ is Hölder (or Lipschitz if $\alpha = 1$) continuous with exponent $\alpha$.

Problem 2 Let $A \subset \mathbb{R}^n$ be non-empty and $f_n : A \rightarrow \mathbb{R}$ a sequence of functions that converge uniformly to a Lipschitz continuous function $f : A \rightarrow \mathbb{R}$. Does it follow that also $f_n$ is Lipschitz continuous for $n$ sufficiently large?

Problem 3 (hard) Let $A \subset \mathbb{R}^k$ be compact and $f_n : A \rightarrow \mathbb{R}$ be a sequence of functions such that

(i) $\{f_n : n \in \mathbb{N}\}$ is equicontinuous, i.e. for any $x \in A$ and any $\varepsilon > 0$ there is $\delta > 0$ such that for any $n \in \mathbb{N}$ and $y \in A$ with $|x - y| < \delta$ it follows that $|f_n(x) - f_n(y)| < \varepsilon$.

(ii) For any $x \in A$ the set $\{f_n(x) : x \in A\} \subset \mathbb{R}$ is bounded.

Show that you can there is a subsequence $(f_{n_l})_{l \in \mathbb{N}}$ that converges uniformly to a continuous function $f : A \rightarrow \mathbb{R}$. Hint: Enumerate the rational points in $A$, then iteratively take subsequences that converge at these points and use a diagonal sequence.

A metric space $(X, d)$ is connected if the only two subsets that are open and closed at the same time are $\emptyset$ and $X$. We call $(X, d)$ path connected if for any two points $x, y \in X$ there is a continuous map $\gamma : [0, 1] \rightarrow X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Problem 4 Let $A \subset [0, 1]$ be a subset that is open and closed.

(i) Show that $[0, 1]$ with the standard metric is connected. Hint: If $A \neq \emptyset$ you may assume wlog that $0 \in A$ (why?), then try to use the supremum of the set
   
   $\{x \in [0, 1] : [0, x] \subseteq A\}$

(ii) Using the first part, show that a path connected space is connected.

(iii) Let $U \subset \mathbb{R}^n$ be open and connected show that $U$ is also path connected. Hint: Fix $x \in U$ and consider the set
   
   $\{y \in U : \exists \gamma \in C^0([0, 1], U) \text{ with } \gamma(0) = x, \gamma(1) = y\}$.

(iv) Let $U \subset \mathbb{R}^n$ be open and connected and $f : U \rightarrow \mathbb{R}$ be differentiable with $Df(x) = 0$ for any $x \in U$. Show that $f$ is constant.

Remark 0.1. The assertion in (iii) really needs the assumption of $U$ being open, see e.g. https://en.wikipedia.org/wiki/Topologist%27s_sine_curve

Problem 5 Show that the Laplacian in spherical coordinates is given by

$$
\Delta f(x, y, z) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial g}{\partial r}(r, \theta, \phi) \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial g}{\partial \theta}(r, \theta, \phi) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 g}{\partial \phi^2}(r, \theta, \phi),
$$

if $(x, y, z) = h(r, \theta, \phi) := (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)$ and $g = f \circ h$.

Problem 6 Let $U, V \subset \mathbb{R}^n$ be open and $\Phi : U \rightarrow V$ a $C^1$-diffeomorphism. Show that

$$
\partial(\Phi(A)) = \Phi(\partial A)
$$

for any $A \subset U$.  

Problem 7 Let $f : \mathbb{R} \to \mathbb{R}$ and write

$$\Gamma_f = \{(x, f(x))\} \subset \mathbb{R}^2$$

for the graph of $f$.

(i) Show that $\Gamma_f$ is a closed subset of $\mathbb{R}^2$ if and only if $f$ is continuous.

(ii) Give an example of a function as above such that $\partial \Gamma_f = \mathbb{R} \times 0 \cup \mathbb{R} \times \{1\}$.

(iii) Show that the graph cannot be an open subset of $\mathbb{R}^2$.

Problem 8 Let $U \subset \mathbb{R}^n$ be open, $x \in U$ and $\xi \in \mathbb{R}^n$ such that $\{x + t\xi : t \in [0, 1]\} \subset U$. Show that for $f : U \to \mathbb{R}$ a continuously differentiable function, we have that

$$f(x + \xi) = \int_0^1 \langle \nabla f(x + t\xi), \xi \rangle \, dt.$$

Problem 9 Let $U \subset \mathbb{R}^n$ be open, $a \in U$ and $f : U \to \mathbb{R}$. Recall the definition of a directional derivative: For $v \in \mathbb{R}^n$

$$D_v f(a) = \lim_{h \to 0} \frac{f(a + hv) - f(a)}{h}.$$

Show the following properties of directional derivatives provided that $f$ is differentiable:

$$D_{v+w} f(a) = D_v f(a) + D_w f(a), \quad D_{\lambda v} f(a) = \lambda D_v f(a),$$

where $v, w \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ (in particular all of these exist). What can you say if you don’t assume that $f$ is differentiable but some/all of the involved directional derivatives exist?