Problem 1
(i) Show that a set \( A \subset \mathbb{R}^n \) has measure zero if and only if for any \( \varepsilon > 0 \) there is a cover by closed rectangles \( (R_i)_{i \in \mathbb{N}} \) such that \( \sum_{i \in \mathbb{N}} \text{vol}(R_i) < \varepsilon \).
(ii) If \( A \subset \mathbb{R}^n \) has measure 0 and \( f: \mathbb{R}^n \to \mathbb{R}^n \) is a Lipschitz function, show that \( f(A) \subset \mathbb{R}^n \) has measure 0.
(iii) For \( r > 0 \) show that \( \{ x \in \mathbb{R}^2 : |x| = r \} \) has measure 0.
(iv) Given an example of an open set \( U \subset \mathbb{R} \) such that the boundary \( \partial U \) does not have measure 0.

Problem 2 Let \( R \subset \mathbb{R}^n \) be a closed rectangle and \( f: R \to \mathbb{R} \) be a bounded function. Let \( S_1, \ldots, S_k \subset R \) be the subrectangles coming from a partition \( P \) of \( R \). Prove that \( f \) is integrable on \( R \) if and only if \( f|_{S_i} \) is integrable for any \( i = 1, \ldots, k \). Moreover, in this case we have that \( \sum_{i=1}^{k} \int_{S_i} f = \int_R f \).

Problem 3 Let \( U \subset \mathbb{R}^n \) be open and \( f: U \to \mathbb{R} \) be bounded. Show that the set \( \mathcal{R} = \{ x \in U : f \text{ is continuous at } x \} \) is a countable intersection of open sets.

Remark 0.1. Using the Baire category theorem one can show that \( \mathbb{Q} \) is not of the type above, hence there can be no function on \( \mathbb{R} \), which is continuous exactly at the rational numbers.

Problem 4 We inductively define sets \( C_n \subset [0, 1] \) as follows \( C_0 = [0, 1] \) and \( C_n = \{ x/3 : x \in C_{n-1} \} \cup \{ 2/3 + x/3 : x \in C_{n-1} \} \) for \( n \geq 1 \). Let \( C = \bigcap_{n=0}^{\infty} C_n \). Show that
(i) \( C \) is compact.
(ii) \( C \) has measure 0.
(iii) \( C \) is uncountable.

For the last part you may use that any real number in \([0, 1] \) can be uniquely written as \( x = \sum_{i=1}^{\infty} a_i 2^{-i} \) with \( a_i \in \{0, 1\} \). (We essentially proved this in last quarters HW.)

Problem 5 Define a function \( f_n: [0, 1] \to \mathbb{R} \) as follows \( f_0(x) = x \)
\[
f_{n+1}(x) = \begin{cases} 
f_n(3x)/2 & \text{if } x \in [0, 1/3] \\
1/2 & \text{if } x \in [1/3, 2/3] \\
1/2 + f_n(3x - 2)/2 & \text{if } x \in [2/3, 1].
\end{cases}
\]

Show that
(i) Each \( f_n \) is continuous.
(ii) \( f_n \) converges uniformly to a function \( f \)
(iii) \( f \) is differentiable in \([0, 1] \setminus C \) with \( f'(x) = 0 \) for any \( x \in [0, 1] \setminus C \).
(iv) \( f(C) = [0, 1] \). Hint: Show that \( f(0) = 0, \, f(1) = 1 \), then apply the intermediate value theorem and use (iii).

Problem 6* Show that \( f \) is Hölder continuous with Hölder exponent \( \alpha = \log(2)/\log(3) \) but not Hölder continuous for any exponent \( \beta > \alpha \).