Problem 1 Show that 
\[ \|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2 \]
for any \( x \in \mathbb{R}^n \). Conclude from this that the set \( \{x \in \mathbb{R}^n : \|x\|_1 = 1\} \) is compact in \((\mathbb{R}^n, \| \cdot \|_1)\).

Problem 2 Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces.
(i) Show that \( f : (X, d_X) \to (Y, d_Y) \) is continuous if and only if \( f^{-1}(O) \subset X \) is open for any open set \( O \subset Y \).
(ii) Let \( f : (X, d_X) \to (Y, d_Y) \) continuous and \( A \subset X \) compact. Show that \( f(A) \subset Y \) is compact.

Problem 3 For the following linear maps find the matrices representing them.
(i) \( F : \mathbb{R}^4 \to \mathbb{R}, (a, b, c, d) \mapsto \int_{-1}^{1} f_{a,b,c,d}(x) \, dx \), where \( f_{a,b,c,d}(x) = ax^3 + bx^2 + cx + dx \)
(ii) \( E : \mathbb{R}^4 \to \mathbb{R}^3, (a, b, c, d) \mapsto (f_{a,b,c,d}(-1), f_{a,b,c,d}(0), f_{a,b,c,d}(1)) \), where \( f_{a,b,c,d} \) is as above

Problem 4 Let \( A \in \mathbb{R}^{n \times m} \) and \( B \in \mathbb{R}^{m \times k} \) be matrices. Denote by \( \phi_A : \mathbb{R}^m \to \mathbb{R}^n \) and \( \phi_B : \mathbb{R}^k \to \mathbb{R}^m \) the associated linear maps. Find the matrix representing the linear map \( \phi_A \circ \phi_B : \mathbb{R}^k \to \mathbb{R}^n \).

A norm on \( \mathbb{R}^n \) is a map \( p : \mathbb{R}^n \to [0, \infty) \) such that
(i) \( p(x) = 0 \) if and only if \( x = 0 \)
(ii) \( p(\lambda x) = |\lambda|p(x) \) for any \( \lambda \in \mathbb{R} \) and \( x \in \mathbb{R}^n \)
(iii) \( p(x + y) \leq p(x) + p(y) \) for any \( x, y \in \mathbb{R}^n \).

Problem 5* Let \( p, q \) be norms on \( \mathbb{R}^n \). Show that there is a constant \( C > 0 \) such that
\[ C^{-1}p(x) \leq q(x) \leq Cp(x) \]
for any \( x \in \mathbb{R}^n \).
Hints: Show that the general case can be deduced from the special case \( p(x) = \|x\|_1 \). For this case, consider the map \( f : S = \{x \in \mathbb{R}^n : \|x\|_1 = 1\} \to \mathbb{R} \) given by \( x \mapsto q(x) \) and show that it is continuous. Then exploit the fact that \( S \) is compact in \((\mathbb{R}^n, \| \cdot \|_1)\) by the first problem and use the way a norm scales.