Problem 1
(a) For symmetry note that if \( x = \lambda y \) for \( \lambda > 0 \) then also \( y = (1/\lambda)x \) with \( 1/\lambda > 0 \). The triangle inequality is done by case distinction. The distance of two points is determined by traveling only along lines through the origin. That’s pretty much exactly like the French railroad system with Paris as the origin.
(b) We claim that the sequence \( a_n = (\cos(1/n), \sin(1/n)) \) is contained in \( B \) but does not have any convergent subsequence. In particular, \( B \) is not compact. Let \( (a_{n_k})_{k \in \mathbb{N}} \) be a subsequence and note that
\[
d(a_{n_k}, a_{n_l}) = 2
\]
if \( n_k \neq n_l \) In particular, \( (a_{n_k}) \) is not Cauchy and thus also not convergent.

Problem 2 Let \( (X, d) \) be a metric space and \( A \subset X \).
(a) (i) Write \( B = \bigcap \{ C \subset X : A \subset C, C \text{ closed} \} \). If \( x \notin B \), there is a closed set \( C \subset X \) with \( A \subset C \) and \( x \notin C \). Since \( X \setminus C \) is open, there is \( \varepsilon > 0 \) such that \( B_\varepsilon(x) \subset X \setminus C \subset X \setminus A \). In particular \( x \notin \overline{A} \). If \( x \notin \overline{A} \), there is \( \varepsilon > 0 \) such that \( B_\varepsilon(x) \subset X \setminus A \). Let \( C = X \setminus B_\varepsilon(x) \). Then \( B \subset C \), hence \( x \notin B \).
(ii) If \( x \in \overline{A} \), there is \( \varepsilon > 0 \) such that \( B_\varepsilon(x) \subset A \). Moreover \( B_\varepsilon(x) \) is open. It follows that \( x \in B_\varepsilon(x) \subset \bigcup \{ O \subset X : O \subset A, O \text{ open} \} \) If \( x \in \bigcup \{ O \subset X : O \subset A, O \text{ open} \} \) there is an open set \( O \subset A \) such that \( x \in O \).
Since \( O \) is open, there has to be \( \varepsilon > 0 \) such that \( B_\varepsilon(x) \subset O \subset A \), hence \( x \in \overline{A} \).
(iii) If \( x \notin \overline{A} \) there is \( \varepsilon > 0 \) such that \( x \in B_\varepsilon(x) \subset A \), hence \( x \in \overline{A} \). If \( x \in A \), then \( x \in B_\varepsilon(x) \cap A \) for any \( \varepsilon > 0 \), hence \( A \subset \overline{A} \). If \( x \notin \overline{A} \) and \( B_\varepsilon(x) \cap X \setminus A \neq \emptyset \) for any \( \varepsilon > 0 \), then \( x \in \partial A \), hence \( \overline{A} \subset \overline{A} \cap \partial A \). If \( x \notin \overline{A} \) and \( B_\varepsilon(x) \cap \partial A \neq \emptyset \) for any \( \varepsilon > 0 \), hence \( \partial A \subset \overline{A} \). If \( x \in \overline{A} \), there is \( \varepsilon > 0 \) such that \( B_\varepsilon(x) \subset A \), in other words \( B_\varepsilon(x) \cap X \setminus A = \emptyset \), hence \( x \notin \partial A \).
(iv) If \( x \notin \overline{A} \), for \( B_{1/n}(x) \cap A \neq \emptyset \) for any \( n \in \mathbb{N} \). Take \( x_n \in A \cap B_{1/n}(x) \). Then \( a_n \in A \) for any \( n \in \mathbb{N} \) and by construction \( d(x_n, x) \leq 1/N \) if \( n \geq N \), hence \( \lim_{n \to \infty} x_n = A \). If conversely \( x : \mathbb{N} \to A \) is a sequence with limit \( x \) and \( \varepsilon > 0 \) is given, there is \( n \in \mathbb{N} \) such that \( d(x_n, x) \leq \varepsilon \). In particular \( A \cap B_\varepsilon(x) \neq \emptyset \).
\( \overline{A} = \{ x \in X : \text{there is a sequence } x : \mathbb{N} \to A \text{ with } \lim_{n \to \infty} x_n = x \} \)
(b) Let \( \phi : \mathbb{N} \to \mathbb{Q} \) be bijective and define \( A_i = \{ \phi(i) \} \). Then \( \bigcup_{i \in \mathbb{N}} A_i = \mathbb{Q} \) and \( \partial \mathbb{Q} = \mathbb{R} \).
On the other hand \( \partial A_i = A_i \) so that \( \bigcup_{i \in \mathbb{N}} \partial A_i = \mathbb{Q} \).

Problem 3
(a) Let \( x, y \in X \) and \( a \in A \), then, by the triangle inequality, we find that
\[
d(x, a) \leq d(x, y) + d(y, a),
\]
which implies
\[
d(x, a) - d(y, a) \leq d(x, y).
\]
By taking the infimum over all \( a \in A \) this gives
\[
d_A(x) - d_A(y) \leq d(x, y)
\]
By symmetry this implies
\[
|d_A(x) - d_A(y)| \leq d(x, y).
\]
(b) If \( x \in \overline{A} \) we have by definition that 
\[
B_\varepsilon(x) \cap A \neq \emptyset
\]
for any \( \varepsilon > 0 \). This implies that 
\[
d_A(x) \leq \varepsilon.
\]
Since \( \varepsilon > 0 \) was arbitrary it follows that \( d_A(x) = 0 \). Suppose now that \( d_A(x) = 0 \) and take \( \varepsilon > 0 \). By the definition of an infimum, there has to be \( a \in A \) such that 
\[
d(x, a) \leq \varepsilon/2
\]
But this implies that \( a \in A \cap B_\varepsilon(x) \). Since \( \varepsilon > 0 \) was arbitrary it follows that \( x \in \overline{A} \)

**Problem 4** Here are two proofs. The first one using sequential compactness, the second one covering compactness. For \( i \in \mathbb{N} \) take \( a_i \in A_i \). Since \( A_i \subset A_1 \) and \( A_1 \) is compact, there is a convergent subsequence \( (a_{i_j}) \), such that 
\[
\lim_{j \to \infty} a_{i_j} = a \in A_1.
\]
Since \( A_{i+1} \subset A_i \) for any \( i \in \mathbb{N} \) we have by construction that \( a_{i_j} \in A_n \) if \( i_j \geq n \). As \( A_n \) is compact, it is in particular closed, hence we also find that 
\[
\lim_{j \to \infty} a_{i_j} \in A_n.
\]
Since this applies to any \( n \in \mathbb{N} \), we obtain that 
\[
a \in \bigcap_{n \in \mathbb{N}} A_n.
\]
Here is the second proof. Suppose that \( \bigcap_{n \in \mathbb{N}} A_n = \emptyset \). In this case, since \( A_n \subset X \) is closed, we have an open cover of \( A_1 \) given by the family \( \{X \setminus A_n\}_{n \in \mathbb{N}} \). Because \( A_1 \) is compact, there is a finite subcover \( \{X \setminus A_n\}_{n=1}^N \). But then 
\[
A_{N+1} \subset A_1 \setminus \bigcup_{n=1}^N (X \setminus A_n) = \emptyset
\]
which is a contradiction.