Problem 1
(a) Since
\[ |1 - f_n(x)| = |x^2/n| \leq 1/n \to 0 \]
if \( x \in [0,1] \) it follows that \( f_n \to 1 \) uniformly.
(b) For \( x \in \mathbb{R} \), if we take some \( N > |x| + 1 \) we find that for any \( n \geq N \) that \( g_n(x) = 0 \). This clearly implies that \( g_n \to 0 \) pointwise. On the other hand, \( g_n(n) = 1 \) so the convergence is not uniform.
(c) If \( x = 0 \), then \( h_n(x) = 0 \) for any \( n \in \mathbb{N} \). If \( x \neq 0 \), we have that \( -n|x| \to -\infty \) as \( n \to \infty \) and hence \( h_n(x) \to 0 \) as \( n \to \infty \). Since any \( h_n \) is continuous but the pointwise limit is not, the convergence can not be uniform.

Problem 2 To show uniform convergence note that the function \( y \mapsto ye^{-y} \) is bounded on \([0, \infty)\) by some constant \( C > 0 \) (we proved this last quarter!). This implies that for \( x \in [0, \infty) \)
\[ |f_n(x)| \leq \frac{1}{n} \left| \frac{x}{n} e^{-x/n} \right| \leq \frac{C}{n}, \]
which easily implies that \( f_n \to 0 \) uniformly. A straightforward computation yields
\[ \int_0^\infty f_n(x) \, dx = 1. \]

Problem 3 You can simply copy the proof of the corresponding statement for continuous functions. Let’s do that. Let \( \varepsilon > 0 \), take \( n \in \mathbb{N} \) such that \( \sup_{x \in K} |f_n(x) - f(x)| \leq \varepsilon \). Since \( f_n \) is uniformly continuous, there is the corresponding \( \delta \). If then \( x, y \in K \) with \( |x - y| < \delta \), we find from the triangle inequality that
\[ |f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(x)| \leq 3\varepsilon. \]

Problem 4 Fix \( x \in [a,b] \), then
\[ |f_n(x) - f_m(x)| \leq \|f_n - f_m\|_{C^0([a,b])}, \]
which implies that \( (f_n(x))_{n \in \mathbb{N}} \) is a Cauchy sequence in \( \mathbb{R} \). As such,
\[ f(x) := \lim_{n \to \infty} f_n(x) \]
exists. We now show that \( (f_n) \) converges uniformly to \( f \), which in particular implies that \( f \) is continuous. Given \( \varepsilon > 0 \) let \( N \in \mathbb{N} \) such that \( \|f_n - f_m\|_{C^0([a,b])} < \varepsilon \) for any \( n, m \geq N \). For \( x \in [a,b] \), take \( m \geq N \) such that \( |f_m(x) - f(x)| \leq \varepsilon \). We then find that
\[ |f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \leq 2\varepsilon. \]
if \( n \geq N \) (the estimate you eventually obtain does not depend on \( x \) or any choice of \( m! \)).

**Problem 5** This goes by the name Dini’s theorem and is a bit tricky. We first make some simplifying assumptions. Since \( f \) is continuous, by subtracting \( f \), we may assume that \( f_n \) converge pointwise monotone to 0.

We argue by contradiction and assume that \( f_n \) does not converge uniformly to \( f \). Then we can find some \( \varepsilon > 0 \) and for any \( N \in \mathbb{N} \) some \( n \geq N \) and a point \( x \in [a, b] \) such that \( f_n(x_n) > \varepsilon \) (the last inequality uses that \( f_n \geq 0 \) and \( f = 0 \)).

If we take \( N = 1 \), this gives us some \( n_1 \geq 1 \) and a corresponding point \( x_1 \in [a, b] \) such that \( f(x_1) > \varepsilon \). Next, take \( N = n_1 + 1 \) to find some \( n_2 > n_1 \) and a corresponding point \( x_2 \in [a, b] \) such that \( f(x_2) > \varepsilon \). By continuing this process inductively, we find a subsequence \( (f_{n_k})_{k \in \mathbb{N}} \) and sequence of points \( (x_k)_{k \in \mathbb{N}} \), all in \([a, b]\), such that \( f_{n_k}(x_k) \geq \varepsilon \). To make our life a bit simpler we simply suppose that \( n_k = k \) (this is only a matter of simplifying notation a bit!). With this notation our choice of the points \( x_n \) reads as follows

\[
f_n(x_n) \geq \varepsilon.
\]

Since the convergence is assumed to be pointwise monotone this implies that

\[
(0.1) \quad f_l(x_n) \geq f_n(x_n) \geq \varepsilon
\]

if \( l \leq n \). By Bolzano–Weierstraß we can find a subsequence of \( (x_n)_{n \in \mathbb{N}} \), which we do not relabel once again, such that

\[
\lim_{n \to \infty} x_n = x_\ast \in [a, b],
\]

where we use that \([a, b]\) is a closed interval. Since \( f_l \) is a continuous function we find that

\[
f_l(x_\ast) = \lim_{n \to \infty} f_l(x_n) \geq \varepsilon,
\]

since when taking the limit we eventually have \( l \leq n \), so that (0.1) applies. Now this works for any \( l \in \mathbb{N} \) and therefore the pointwise convergence of \( f_l \) implies that

\[
0 \lim_{l \to \infty} f_l(x_\ast) \geq \varepsilon
\]

which is a contradiction.