Problem 1 Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that for any $k \in \mathbb{N}$ there is a polynomial $p_k$, such that

$$f^{(k)}(x) = \begin{cases} p_k(1/x)f(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Problem 2 Show that the improper integral

$$\int_0^{\infty} x^{-1/2}e^{-x} \, dx$$

exists. (Note that the integrand is unbounded near 0.)

Problem 3* Let $f, g: [a, b] \to \mathbb{R}$ be integrable functions such that $f(x) = g(x)$ for any $x \in \mathbb{Q}$. Show that

$$\int_a^b f = \int_a^b g.$$