Problem 1  Since $g$ is integrable, there are $c < d$ such that $g : [a, b] \rightarrow [c, d]$. As we have proved last quarter, the continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ is uniformly continuous on $[c, d]$. Therefore, for $\varepsilon > 0$ given, we can choose $\delta > 0$ such that we have

$$|f(x) - f(y)| \leq \varepsilon$$

for $x, y \in [c, d]$ with $|x - y| \leq \delta$. Moreover, we may of course assume that $\delta \leq \varepsilon$. Since $g$ is integrable, we can choose step functions $\phi, \psi \in S(a, b)$ with $\phi \leq f \leq \psi$ and

$$\int_a^b (\psi - \phi) \leq \delta^2.$$

Let $a = t_0 < t_1 < \cdots < t_{k-1} < t_k = b$ be a partition of $[a, b]$ such that $\phi$ and $\psi$ are constant on all of the subintervals $(t_i, t_{i+1})$. We define new step functions $\tilde{\phi}, \tilde{\psi} \in S(a, b)$ by

$$\tilde{\psi} = \sup_{(t_i, t_{i+1})} (f \circ g) \text{ on } [t_i, t_{i+1})$$

$$\tilde{\phi} = \inf_{(t_i, t_{i+1})} (f \circ g) \text{ on } [t_i, t_{i+1}).$$

Clearly, we have $\tilde{\phi} \leq f \circ g \leq \tilde{\psi}$. Note that if $\psi - \phi \leq \delta$ on $(t_i, t_{i+1})$ then $\tilde{\psi} - \tilde{\phi} \leq \varepsilon$ on $(t_i, t_{i+1})$, so we should try to estimate the total length of the intervals on which this can fail, let’s call them the bad intervals. In order to do so, let use denote by $l$ be the sum of the lengths of the bad intervals. We have that

$$l \delta \leq \int_a^b (\psi - \phi) \leq \delta^2,$$

which implies that

$$l \leq \delta.$$

This in turn easily gives that

$$\int_a^b (\tilde{\psi} - \tilde{\phi}) \leq ((b - a) - l)\varepsilon + \sup_{[c,d]} |f| l \leq (b - a)\varepsilon + \sup_{[c,d]} |f| \delta \leq ((b - a) + \sup_{[c,d]} |f|)\varepsilon.$$

Since $\varepsilon$ was arbitrary, this shows that $f \circ g : [a, b] \rightarrow \mathbb{R}$ is integrable.

Problem 2  The first part is induction. The base case is exactly the definition of convexity. For the induction step we may permute the $\lambda_i$ if necessary and assume that $\lambda_{n+1} \neq 0$. Then we can write

$$\sum_{i=1}^{n+1} \lambda_i x_i = \sum_{i=1}^{n} \lambda_i x_i + \lambda_{n+1} x_{n+1}$$

$$= (1 - \lambda_{n+1}) \sum_{i=1}^{n} \frac{\lambda_i}{1 - \lambda_{n+1}} x_i + \lambda_{n+1} x_{n+1}.$$
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Observe that
\[ \sum_{i=1}^{n} \frac{\lambda_i}{1 - \lambda_{n+1}} = 1 \]
by assumption. Therefore, since \( f \) is convex and by induction hypothesis, this implies that
\[
f\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) = f\left((1 - \lambda_{n+1}) \sum_{i=1}^{n} \frac{\lambda_i}{1 - \lambda_{n+1}} x_i + \lambda_{n+1} x_{n+1}\right) \\
\leq (1 - \lambda_{n+1}) f\left(\sum_{i=1}^{n} \frac{\lambda_i}{1 - \lambda_{n+1}} x_i\right) + \lambda_{n+1} f(x_{n+1}) \\
\leq (1 - \lambda_{n+1}) \sum_{i=1}^{n} \frac{\lambda_i}{1 - \lambda_{n+1}} f(x_i) + \lambda_{n+1} f(x_{n+1}) \\
\leq \sum_{i=1}^{n+1} \lambda_i f(x_i).
\]

The second part follows by approximating the integral by Riemann sums. Since \( f \) and \( f \circ g \) are integrable we may simply use our favourite Riemann sum to approximate them. By the first part (note that the normalization factor \( 1/(b - a) \) allows us to apply this) we have that
\[
f\left(\frac{1}{b - a} \sum_{i=1}^{n} g\left(a + i \frac{b - a}{n}\right) \frac{b - a}{n}\right) \leq \frac{1}{b - a} \sum_{i=1}^{n} (f \circ g)\left(a + i \frac{b - a}{n}\right) \frac{b - a}{n}
\]
By the theorem on Riemann sums and since \( f \) is continuous passing to the limit \( n \to \infty \) gives
\[
f\left(\frac{1}{b - a} \int_{a}^{b} g(x) \, dx\right) = f\left(\lim_{n \to \infty} \frac{1}{b - a} \sum_{i=1}^{n} g\left(a + i \frac{b - a}{n}\right) \frac{b - a}{n}\right) \\
= \lim_{n \to \infty} f\left(\frac{1}{b - a} \sum_{i=1}^{n} g\left(a + i \frac{b - a}{n}\right) \frac{b - a}{n}\right) \\
\leq \lim_{n \to \infty} \frac{1}{b - a} \sum_{i=1}^{n} (f \circ g)\left(a + i \frac{b - a}{n}\right) \frac{b - a}{n} \\
= \frac{1}{b - a} \int_{a}^{b} (f \circ g)(x) \, dx.
\]

**Problem 3** The first two parts are done by integration by parts. We have for given \( x \in \mathbb{R} \) that
\[
\int_{0}^{x} t \sin(t) \, dt = -t \cos(t)\big|_{t=0}^{x} + \int_{0}^{x} \cos(t) \, dt \\
= -x \cos(x) + \sin(x),
\]
which defines a primitive on all of \( \mathbb{R} \). Using the first part we find
\[
\int_0^x t^2 \cos(t) \, dt = t^2 \sin(t){\bigg |}_0^x - 2 \int_0^x t \sin(t) \, dt
\]
\[
= x^2 \sin(x) + 2x \cos(x) - 2 \sin(x),
\]
which is also a primitive on all of \( \mathbb{R} \).

For the third part we use the substitution \( s = \sin(t) \) and get for \( x \in [-1, 1] \) that
\[
\int_0^x \arcsin(t) \, dt = \int_0^{\arcsin(x)} s \cos(s) \, ds
\]
Analogously to the first part, one finds that \( s \sin(s) + \cos(s) \) is a primitive of \( s \cos(s) \). Therefore, we get
\[
\int_0^x \arcsin(t) \, dt = s \sin(s) + \cos(s){\bigg |}_{s=0}^{\arcsin(x)}
\]
\[
= x \arcsin(x) + \cos(\arcsin(x))
\]
\[
= x \arcsin(x) + \sqrt{1 - x^2}.
\]

For \( n, m \in \mathbb{N} \) we integrate by parts (using that \( \sin(nx) \) and \( \sin(mx) \) vanish at 0 and \( 2\pi \)) twice to find that
\[
\int_0^{2\pi} \sin(nx) \sin(mx) \, dx = -\frac{m}{n} \int_0^{2\pi} \cos(nx) \cos(mx) \, dx
\]
\[
= \frac{m^2}{n^2} \int_0^{2\pi} \sin(nx) \sin(mx) \, dx
\]
This implies that if \( n \neq m \), then
\[
\int_0^{2\pi} \sin(nx) \sin(mx) \, dx = \int_0^{2\pi} \cos(nx) \cos(mx) \, dx = 0.
\]
For \( n = m \), we use that (by symmetry considerations), we need to have
\[
\int_0^{2\pi} \cos^2(nx) \, dx = \int_0^{2\pi} \sin^2(nx) \, dx.
\]
We then combine this with
\[
2\pi = \int_0^{2\pi} \left( \sin^2(nx) + \cos^2(nx) \right) \, dx
\]
to find
\[
\int_0^{2\pi} \cos^2(nx) \, dx = \int_0^{2\pi} \sin^2(nx) \, dx = \pi.
\]

**Problem 4** Why should you expect this to hold? On the one hand, the rescaling of \( g \) essentially lets you see an entire copy of \( g \) within an interval of size \( 1/n \). On the other hand, since \( f \) is (uniformly) continuous, once \( n \) is sufficiently large, \( f \) is almost constant on any interval of size \( 1/n \). Finally, combine this with the observation that the assertion trivially holds if \( f \) is constant.
If we assume $g \geq 0$ there is a short proof as follows. By the mean value theorem, we find $\xi_i \in [i/n, (i + 1)/n]$ such that

$$
\int_0^1 f(x)g(nx) \, dx = \sum_{i=0}^{n-1} \int_{i/n}^{(i+1)/n} f(x)g(nx) \, dx \\
= \sum_{i=0}^{n-1} \int_{i/n}^{(i+1)/n} f(x) \, dx \\
= \sum_{i=0}^{n-1} f(\xi_i) \int_{i/n}^{(i+1)/n} g(nx) \, dx \\
= \sum_{i=0}^{n-1} f(\xi_i) \frac{1}{n} \int_{i}^{i+1} g(x) \, dx \\
= \int_0^1 g(x) \, dx \sum_{i=0}^{n-1} f(\xi_i) \frac{1}{n},
$$

where we have used the substitution formula and the periodicity of $g$. By the theorem on Riemann sums, the last term converges to

$$
\left( \int_0^1 g(x) \, dx \right) \left( \int_0^1 f(x) \, dx \right)
$$

for $n \to \infty$.

**Problem 5** We can not use the mean value theorem anymore. Therefore, the computation above only works up to the second line. Instead of the mean value theorem, we will use that $f$ is uniformly continuous, which will essentially allow us to pretend that $f$ is constant on the short subintervals. We start as above and find that

$$
\int_0^1 f(x)g(nx) \, dx = \sum_{i=0}^{n-1} \int_{i/n}^{(i+1)/n} f(x)g(nx) \, dx \\
= \sum_{i=0}^{n-1} \int_{i/n}^{(i+1)/n} f(x) \, dx \\
= \sum_{i=0}^{n-1} \int_{i/n}^{(i+1)/n} (f(x) - f(i/n))g(nx) \, dx + \sum_{i=0}^{n-1} \int_{i/n}^{(i+1)/n} f(i/n)g(nx) \, dx \\
= \sum_{i=0}^{n-1} \int_{i/n}^{(i+1)/n} (f(x) - f(i/n))g(nx) \, dx + \int_0^1 g(x) \, dx \sum_{i=0}^{n-1} f(i)/n \frac{1}{n}
$$
by the same computation as above. Again by the theorem on Riemann sums, the second summand converges to
\[
\left( \int_0^1 g(x) \, dx \right) \left( \int_0^1 f(x) \, dx \right)
\]
for \( n \to \infty \). Therefore, we need to show that the first summand goes to zero. Since \( f \) is continuous on \([0,1]\) it is uniformly continuous on \([0,1]\). Thus, given \( \varepsilon > 0 \), we can choose \( N \in \mathbb{N} \) such that if \( x, y \in [0,1] \) with \( |x - y| \leq 1/N \), then \( |f(x) - f(y)| < \varepsilon \). For \( n \geq N \), we obtain from the triangle inequality that
\[
\left| \sum_{i=0}^{n-1} \int_{i/n}^{(i+1)/n} (f(x) - f(i/n)) g(nx) \, dx \right| \leq \sum_{i=0}^{n-1} \int_{i/n}^{(i+1)/n} |f(x) - f(i/n)||g(nx)| \, dx
\]
\[
\leq \varepsilon \sum_{i=0}^{n-1} \int_{i/n}^{(i+1)/n} |g(nx)| \, dx
\]
\[
\leq \varepsilon \int_0^1 |g(nx)| \, dx
\]
\[
\leq \sup_{[0,1]} |g| \varepsilon,
\]
where we have used the periodicity of \( g \). Since \( g \) is bounded, the assertion follows.