Problem 1 (Fundamental Lemma of the Calculus of Variations) Let \( f : [a, b] \to \mathbb{R} \) be continuous such that
\[
\int_a^b f(x)g(x)\,dx = 0
\]
for any continuous function \( g : [a, b] \to \mathbb{R} \) with \( g(a) = g(b) = 0 \). Show that \( f = 0 \).

Problem 2 Let \( f : [1, 2] \to \mathbb{R} \) be given by
\[
f(x) = \begin{cases} 
1/q & \text{if } x = p/q \text{ with } p, q \in \mathbb{N} \text{ coprime} \\
0 & \text{if } x \notin \mathbb{Q}.
\end{cases}
\]
Show that \( f \) is integrable and compute its integral.

Problem 3 Let \( f : [a, b] \to \mathbb{R} \) be integrable.

(a) Show that there is \( c \in [a, b] \) such that \( f \) is continuous at \( c \).

(b) Show that the set \( C := \{ x \in [a, b] : f \text{ is continuous at } x \} \) is dense in \( [a, b] \).

Hints: For part (a) try to use step functions \( \phi_n \leq f \leq \psi_n \) with
\[
\int_a^b (\psi_n - \phi_n) \leq 1/n \text{ and the nested interval theorem. For part (b) use part (a).}
\]

Problem 4 (HW 1 - 2 continued) For \( a < b \), let \( f \in C^0([a, b]), g \in C^1([a, b]) \) monotone, and \( p : [a, b] \to [0, \infty) \) integrable. Prove that

(a) There is \( \xi \in [a, b] \) such that
\[
\int_a^b f(x)p(x)\,dx = f(\xi) \int_a^b p(x)\,dx.
\]

Hint: Assume first that \( \int_a^b p(x)\,dx = 1 \). Also, you may use without proof that the product of two integrable functions is integrable.

(b) There is \( \xi \in [a, b] \) such that
\[
\int_a^b f(x)g(x)\,dx = g(a) \int_a^\xi f(x)\,dx + g(b) \int_\xi^b f(x)\,dx.
\]

Hint: Integration by parts and part (a).