Math 161 (41) - Midterm Exam 2

Fall 2018
Monday November 19, 2018 - 11:30 to 12:20

- Read everything carefully.
- Write readable.
- Write your answers on the page of the problem. If you need additional space, use the opposite side.
- Give full proofs of all your statements, indicate if you use a result from class or homework.
- If a problem has multiple parts, you may assume any part for all succeeding parts.
- In the first problem no proofs are required.
- There are four problems.
- You are not allowed to use calculators, books or notes.

Good Luck!

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Problem 1. (3+3+3+3+3 points) Decide for each of the following statements if they are true or false:

(i) The function $f: \{1/n : n \in \mathbb{N}\} \cup \{0\} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1/n^2 & \text{if } x = 1/n \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous.

True [ ] False [ ]

(ii) The least upper bound of the set $\{x \in \mathbb{Q} : x^2 < 9\}$ is a rational number.

True [ ] False [ ]

(iii) Let $A, B \subset \mathbb{R}$ be bounded and non-empty. The least upper bound of $A - B = \{a - b : a \in A, b \in B\}$ satisfies

$$\sup(A - B) = \sup(A) - \inf(B).$$

True [ ] False [ ]

(iv) Let $f: [a, b) \rightarrow \mathbb{R}$ be continuous, then $f$ attains its maximum.

True [ ] False [ ]

(v) Let $f: [a, b) \rightarrow \mathbb{R}$ be uniformly continuous, then $f$ is bounded.

True [ ] False [ ]
Problem 2. \((4+5+1\text{ points})\)

(i) State the intermediate value theorem

Let now \(f: [0, 2] \to \mathbb{R}\) be a continuous function with \(f(0) = f(2)\). Show that

(ii) The function \(g: [0, 1] \to \mathbb{R}\) defined by \(g(x) = f(x + 1) - f(x)\) has a zero.

(iii) There is a point \(y \in [0, 1]\) such that \(f(y) = f(y+1)\).
Problem 3. (4+5+4 points)

(i) State the definition of a uniformly continuous function.

(ii) Let \( \rho : [0, \infty) \to [0, \infty) \) be a continuous function with \( \rho(0) = 0 \) and let \( I \subset \mathbb{R} \). Show that a function \( f : I \to \mathbb{R} \) with

\[
|f(x) - f(y)| \leq \rho(|x - y|) \quad (*)
\]

is uniformly continuous.

(iii) Let \( I = [0, 1] \). For any continuous function \( \rho : [0, \infty) \to [0, \infty) \) with \( \rho(0) = 0 \) find a uniformly continuous function \( f : I \to \mathbb{R} \) such that \( (*) \) does not hold.
Problem 4. (4+4+4 points)

(i) State the definition of a function being differentiable in a point \(a\).

(ii) Show that
\[
\lim_{x \to 0} \sqrt{|x|} = 0.
\]

(iii) Let \(g: \mathbb{R} \to \mathbb{R}\) be a function such that \(|g(x)| \leq 1/\sqrt{|x|}\) for any \(x \neq 0\). Show that the function \(f: \mathbb{R} \to \mathbb{R}\) defined by
\[
f(x) = \begin{cases} 
  x^2 g(x) & \text{if } x \neq 0 \\
  0 & \text{if } x = 0 
\end{cases}
\]
is differentiable at \(0\).