Math 161 (41) - Midterm Exam 1

Fall 2018
Monday October 22, 2018 - 11:30 to 12:20

- Read everything carefully.
- Write readable.
- Write your answers on the page of the problem. If you need additional space, use the opposite side.
- Give full proofs of all your statements, indicate if you use a result from class or homework.
- If a problem has multiple parts, you may assume any part for all succeeding parts.
- In the first problem no proofs are required.
- There are five problems.
- You are not allowed to use calculators, books or notes.

*Good Luck!*

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Problem 1. (2+2+2+2+2 points) Decide for each of the following statements if they are true or false:

(i) \( \{0, \{0\}\} \neq \{0\} \).

True  [ ]
False [ ]

(ii) The integers \( \mathbb{Z} \) with the usual addition and multiplication are a field.

True  [ ]
False [ ]

(iii) Every non-empty subset \( A \subseteq \mathbb{N} \) has a smallest element.

True  [ ]
False [ ]

(iv) The set \( \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 4\} \) defines a function \( f : \mathbb{R} \to \mathbb{R} \).

True  [ ]
False [ ]

(v) Let \( f, g : \mathbb{R} \to (0, \infty) \) be functions and \( a \in \mathbb{R} \) such that \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) do not exist. Then also \( \lim_{x \to a} \frac{f(x)}{g(x)} \) does not exist.

True  [ ]
False [ ]

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Problem 2. (3+3+4 points) Recall that a field is a set $F$ together with maps $+ : F \times F \to F$ and $\cdot : F \times F \to F$ such that the following properties hold

(A1) $a + b = b + a$, for any $a, b \in F$,

(A2) $(a + b) + c = a + (b + c)$, for any $a, b, c \in F$,

(A3) there is an element $0 \in F$ such that $a + 0 = a$, for any $a \in F$,

(A4) for any $a \in F$ there is $b \in F$ such that $a + b = 0$,

(M1) $a \cdot b = b \cdot a$, for any $a, b \in F$,

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, for any $a, b, c \in F$,

(M3) there is an element $1 \in F$ such that $1 \neq 0$ and $1 \cdot a = a$ for any $a \in F$,

(M4) for any $a \in F$ with $a \neq 0$ there is $b \in F$ such that $a \cdot b = 1$,

(D) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$, for any $a, b, c \in F$.

Prove the following statements from the definition of a field $(F, +, \cdot)$ and indicate in each step, which property you use.

(i) If $a \in F$ and $b, c \in F$ such that $a + b = 0$ and $a + c = 0$, then $b = c$

From here on you may use in addition, that $0 \cdot a = 0$ for any $a \in F$.

(ii) For any $a, b \in F$, we have that $-(a \cdot b) = (-a) \cdot b$.

(iii) For any $a, b \in F$, we have that $a \cdot b = (-a) \cdot (-b)$. 
Problem 3. (10 points) For $n \in \mathbb{N}$, show by induction that
\[
\sum_{k=1}^{n} (2k - 1)^2 = \frac{n(2n + 1)(2n - 1)}{3}.
\]
Problem 4. (4+6 points)

(i) State the definition of the limit of a function $f$ at a point $a \in \mathbb{R}$.

(ii) Show from the definition of a limit that

$$\lim_{x \to a} \left( 3x + \frac{1}{x^2 + 1} \right) = 3a + \frac{1}{a^2 + 1}$$

for any $a \in \mathbb{R}$. 
Problem 5. (2+4+4 points) Define the function $s: \mathbb{R} \to \mathbb{R}$ by
\[ s(x) = x - |x - 2k| \text{ for } x \in (2k - 1, 2k + 1], \ k \in \mathbb{Z}. \]

(i) Sketch the graph of $s$.
(ii) Show that $|s(x) - x| \leq 1$ for any $x \in \mathbb{R}$.
(iii) Does $\lim_{x \to 0} xs(1/x)$ exist? If it exists, what is its value?