• Read everything carefully.
• Write readable.
• Write your answers on the page of the problem. If you need additional space, use the opposite side.
• Give full proofs of all your statements, indicate if you use a result from class or homework.
• If a problem has multiple parts, you may assume any part for all succeeding parts.
• In the first problem no proofs are required.
• There are four problems.
• You are not allowed to use calculators, books or notes.

*Good Luck!*

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>/10</td>
</tr>
<tr>
<td>2</td>
<td>/10</td>
</tr>
<tr>
<td>3</td>
<td>/10</td>
</tr>
<tr>
<td>4</td>
<td>/10</td>
</tr>
<tr>
<td>Total</td>
<td>/40</td>
</tr>
</tbody>
</table>
Problem 1. (3+3+3+3+3 points) Decide for each of the following statements if they are true or false:

(i) The function \( f : \{1/n : n \in \mathbb{N}\} \rightarrow \mathbb{R} \) defined by \( f(1/n) = n \) is continuous.

<table>
<thead>
<tr>
<th>True</th>
<th>False</th>
</tr>
</thead>
<tbody>
<tr>
<td>☐</td>
<td>☐</td>
</tr>
</tbody>
</table>

(ii) The least upper bound of the set \( \{x \in \mathbb{Q} : x^2 < 2\} \) is a rational number.

<table>
<thead>
<tr>
<th>True</th>
<th>False</th>
</tr>
</thead>
<tbody>
<tr>
<td>☐</td>
<td>☐</td>
</tr>
</tbody>
</table>

(iii) Let \( A, B \subset [0, \infty) \) be non-empty and bounded from above. The set \( A \cdot B = \{a \cdot b : a \in A, b \in B\} \) has \( \sup(A \cdot B) = \sup(A) \cdot \sup(B) \).

<table>
<thead>
<tr>
<th>True</th>
<th>False</th>
</tr>
</thead>
<tbody>
<tr>
<td>☐</td>
<td>☐</td>
</tr>
</tbody>
</table>

(iv) Let \( f : [a, b) \rightarrow \mathbb{R} \) be continuous, then \( f \) is bounded.

<table>
<thead>
<tr>
<th>True</th>
<th>False</th>
</tr>
</thead>
<tbody>
<tr>
<td>☐</td>
<td>☐</td>
</tr>
</tbody>
</table>

(v) Let \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) be functions and \( a \in \mathbb{R} \). If \( f(a) \neq 0 \), the function \( f \) is differentiable in \( a \) and \( g \) is not differentiable in \( a \), then also \( f \cdot g \) is not differentiable in \( a \).

<table>
<thead>
<tr>
<th>True</th>
<th>False</th>
</tr>
</thead>
<tbody>
<tr>
<td>☐</td>
<td>☐</td>
</tr>
</tbody>
</table>

Solution

(i) True: The set \( A = \{1/n : n \in \mathbb{N}\} \) is discrete, i.e. for any \( x \in A \) there is \( \delta > 0 \) such that \( (a - \delta, a + \delta) \cap A = \{a\} \).

(ii) False: \( \sqrt{2} \notin \mathbb{Q} \).

(iii) True: Same proof as for \( A + B \) works under the assumption \( A, B \subset [0, \infty) \).

(iv) False: Consider e.g. \( f : [-1, 0) \rightarrow \mathbb{R}, x \mapsto 1/x \).

(v) True: If \( f \cdot g \) were differentiable in \( a \), then also \( g = \frac{f}{f} \) would be by the quotient rule, since \( f(a) \neq 0 \).
Problem 2. (4+6 points)

(i) State the intermediate value theorem.

(ii) Let $a < b$ and $f, g : [a, b] \rightarrow \mathbb{R}$ continuous functions with $f(a) < g(a)$ and $f(b) > g(b)$. Show that there is $x \in (a, b)$ such that $f(x) = g(x)$.

Solution

(i) For $a < b$ let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. If $f(a) < f(b)$ and $d \in \mathbb{R}$ with $f(a) < d < f(b)$. Then there is $c \in (a, b)$ such that $f(c) = d$.

(ii) Define $h : [a, b] \rightarrow \mathbb{R}$ by $h(x) = f(x) - g(x)$. Since $f$ is the sum of two continuous functions, it is also continuous. By assumption, we find that $h(a) = f(a) - g(a) < 0$ and $h(b) = f(b) - g(b) > 0$. Therefore, by the intermediate value theorem, there is $x \in (a, b)$ such that $0 = h(x) = f(x) - g(x)$, that is $f(x) = g(x)$. 

Problem 3. (4+11 points)

(i) State the definition of a uniformly continuous function.

(ii) Let \( f : [0, \infty) \to \mathbb{R} \) be a continuous function such that
\[
\lim_{x \to \infty} f(x) = c \text{ for some } c \in \mathbb{R}, \text{ i.e. for any } \varepsilon > 0 \text{ there is } C > 0 \text{ such that } x > C \text{ implies that } |f(x) - c| < \varepsilon.
\]
Show that \( f \) is uniformly continuous.

Solution

(i) Let \( I \subset \mathbb{R} \). A function \( f : I \to \mathbb{R} \) is called uniformly continuous if for any \( \varepsilon > 0 \) there is some \( \delta > 0 \) such that if \( x, y \in I \) with \(|x - y| < \delta\) then \(|f(x) - f(y)| < \varepsilon\).

(ii) Let \( \varepsilon > 0 \) be given. We choose \( C > 0 \) such that if \( z > C \), then \(|f(z) - l| < \varepsilon/2\). Then, if \( x, y > C \) we get from the triangle inequality that
\[
|f(x) - f(y)| \leq |f(x) - l| + |l - f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

Since \( f \) is continuous on \([0, C + 2]\), it is also uniformly continuous on \([0, C + 2]\) as proved in class. Therefore, there is some \( \delta_0 \), such that if \( x, y \in [0, C + 2] \) with \(|x - y| < \delta_0\), then \(|f(x) - f(y)| < \varepsilon\). We now define \( \delta = \min(1, \delta_0) \). If \( x \leq C + 1 \) and \(|x - y| \leq \delta \) it follows that \( y \leq C + 1 + 1 = C + 2 \) so that \(|f(x) - f(y)| < \varepsilon\) by the choice of \( \delta \). If \( x \geq C + 1 \) and \(|x - y| < \delta\), then \( y > C + 1 - 1 = C \) and \(|f(x) - f(y)| < \varepsilon\) by the computation above and the choice of \( \delta \).
Problem 4. (4+6 points)

(i) State the definition of a function being differentiable in a point a.

(ii) Let \( g: \mathbb{R} \to \mathbb{R} \) be bounded, i.e. there is some \( M > 0 \) such that \( |g(x)| \leq M \) for any \( x \in \mathbb{R} \). Show that \( f: \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x^2 g(x) \) is differentiable at 0.

Solution

(i) Let \( f: I \to \mathbb{R} \) for an open interval \( I \) with \( a \in I \). We call \( f \) differentiable at \( a \) if the limit

\[
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

exists.

(ii) We have

\[
\left| \frac{f(h) - f(0)}{h} \right| = \left| \frac{h^2 g(x)}{h} \right| \leq M|h|
\]

In particular, we find that

\[
\lim_{h \to 0} \frac{h^2 g(x)}{h} = 0.
\]

(You can use the sandwich lemma here if you like. But it’s also immediate from the definition if you choose \( \delta = \varepsilon/M \).)