Math 161 (21) - Midterm Exam 2

Fall 2018
Monday November 19, 2018 - 9:30 to 10:20

• Read everything carefully.
• Write readable.
• Write your answers on the page of the problem. If you need additional space, use the opposite side.
• Give full proofs of all your statements, indicate if you use a result from class or homework.
• If a problem has multiple parts, you may assume any part for all succeeding parts.
• In the first problem no proofs are required.
• There are four problems.
• You are not allowed to use calculators, books or notes.

Good Luck!

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<td>1</td>
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Problem 1. (3+3+3+3+3 points) Decide for each of the following statements if they are true or false:

(i) The function \( f : \{1/n : n \in \mathbb{N}\} \to \mathbb{R} \) defined by \( f(1/n) = n \) is continuous.
   - True [ ]
   - False [ ]

(ii) The least upper bound of the set \( \{x \in \mathbb{Q} : x^2 < 2\} \) is a rational number.
   - True [ ]
   - False [ ]

(iii) Let \( A, B \subset [0, \infty) \) be non-empty and bounded from above. The set \( A \cdot B = \{a \cdot b : a \in A, b \in B\} \) has \( \sup(A \cdot B) = \sup(A) \cdot \sup(B) \).
   - True [ ]
   - False [ ]

(iv) Let \( f : [a, b) \to \mathbb{R} \) be continuous, then \( f \) is bounded.
   - True [ ]
   - False [ ]

(v) Let \( f, g : \mathbb{R} \to \mathbb{R} \) be functions and \( a \in \mathbb{R} \). If \( f(a) \neq 0 \), the function \( f \) is differentiable in \( a \), and \( g \) is not differentiable in \( a \), then also \( f \cdot g \) is not differentiable in \( a \).
   - True [ ]
   - False [ ]
Problem 2. (4+6 points)

(i) State the intermediate value theorem.

(ii) Let $a < b$ and $f, g: [a, b] \rightarrow \mathbb{R}$ continuous functions with $f(a) < g(a)$ and $f(b) > g(b)$. Show that there is $x \in (a, b)$ such that $f(x) = g(x)$. 
Problem 3. (4+11 points)

(i) State the definition of a uniformly continuous function.

(ii) Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that
\[ \lim_{x \to \infty} f(x) = l \text{ for some } l \in \mathbb{R}, \text{ i.e. for any } \varepsilon > 0 \text{ there is } C > 0 \text{ such that } x > C \implies |f(x) - l| < \varepsilon. \] Show that $f$ is uniformly continuous.
Problem 4. (4+6 points)

(i) State the definition of a function being differentiable in a point $a$.

(ii) Let $g: \mathbb{R} \to \mathbb{R}$ be bounded, i.e. there is some $M > 0$ such that $|g(x)| \leq M$ for any $x \in \mathbb{R}$. Show that $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 g(x)$ is differentiable at $0$. 