Problem 1 Let $I \subset \mathbb{R}$ be an open interval, $a \in I$ and $f : I \to \mathbb{R}$ two times differentiable at $a$. Prove that

$$f''(a) = \lim_{h \to 0} \frac{f(a + h) - 2f(a) + f(a - h)}{h^2}. $$

Hint: Consider first the case $f'(a) = f''(a) = 0$ and reduce the general case to this case. In the former case try to use the mean value theorem.

Problem 2 Let $f : (0, \infty) \to \mathbb{R}$ be differentiable with $\lim_{x \to \infty} f(x) = l \in \mathbb{R}$ and $\lim_{x \to \infty} f'(x) = l' \in \mathbb{R}$. Show that this implies $l' = 0$.

Problem 3 Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Assume that there is a constant $C > 0$ such that $|f'(x)| \leq C$ for any $x \in \mathbb{R}$. Show that $f$ has to be uniformly continuous.

Problem 4

(i) Let $a < b$ and let $f, g : [a, b] \to \mathbb{R}$ be continuous functions, which are differentiable in $(a, b)$. If $f(a) \leq g(a)$ and $f'(x) \leq g'(x)$ for any $x \in (a, b)$, show that $f \leq g$ everywhere on $[a, b]$.

(ii) Let $f : I \to \mathbb{R}$ be differentiable, where $I$ is an open interval. If $[a, b] \subset I$ and $f'(a) < c < f'(b)$, show that there is $\xi \in (a, b)$ such that $f'(\xi) = c$. Note that we do not assume that $f$ is continuously differentiable!

Problem 5*

(i) Let $(F, +, \cdot)$ be a field. Assume that $1 + 1 \neq 0$. Show that for $a, b \in F$ with $b - a = a - b$, it follows that $a = b$.

(ii) Show that this does not hold in general if we do not assume $1 + 1 \neq 0$. Hint: Try to construct a field with exactly 4 elements.

Problem 6*

Show that for any $n \in \mathbb{N}$, we have that

$$\sum_{j=1}^{n} \frac{1}{\sqrt{j}} \leq 2\sqrt{n} - 1.$$

Problem 7*

(i) Decide (with proof) for the set $M$ given below if infimum, supremum, maximum, and minimum exist and determine them if appropriate:

$$M = \{1 - 2^{-n} + 3^{-m} : n, m \in \mathbb{N}\}.$$ 

(ii) Let $A \subset \mathbb{R}$ be non-empty and bounded from above. Show that $-A := \{-a : a \in A\}$ is non-empty and bounded from below. Moreover, $\inf(-A) = -\sup(A)$. 

Problem 8* Prove from the definition of a limit that
(i) \( \lim_{x \to 0} \sqrt[3]{x} = 0 \)
(ii) \( \lim_{x \to a} \sqrt[3]{x} = \sqrt[3]{a} \) if \( a \neq 0 \).
(iii) \( \lim_{x \to a} \left(\frac{x}{x^2} + 2x^3\right) = \frac{a}{a^2} + 2a^3 \) if \( a \neq 2 \).

Problem 9* Determine (with proof) for any of the following functions, where it is continuous.
(i) \( f : \mathbb{Q} \to \mathbb{R}, f(x) = \frac{1}{q} \) if \( x = \frac{p}{q} \) with \( p, q \in \mathbb{Z} \) coprime.
(ii) For \( n \in \mathbb{N} \) let \( A_n = \{ x \in \mathbb{Q} : x = \frac{p}{q}, p, q \in \mathbb{Z}, |q| \leq n \} \) and let \( g : A_n \to \mathbb{R}, x \mapsto f(x) \).

Problem 10* Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function, such that \( f(x) \in \mathbb{Q} \) for any \( x \in \mathbb{R} \). Show that \( f \) has to be constant.

Problem 11* Let \( f, g : \mathbb{R} \to \mathbb{R} \) be two functions.
(i) Show that if \( g \) is bounded and \( f \) is differentiable at 0 with \( f(0) = f'(0) = 0 \), then also \( f \cdot g \) is differentiable at 0.
(ii) Show that if \( \lim_{x \to 0} xg(x) = 0 \) and \( f \) is two times differentiable with \( f(0) = f'(0) = 0 \) and \( f'' \) is bounded, then also \( f \cdot g \) is differentiable at 0.

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