Problem 1 (a) First note that the set \((a-\delta, a+\delta) \cap \text{dom}(|f|) \setminus \{a\}\) is non-empty for any \(\delta > 0\), since \(\text{dom}(|f|) = \text{dom}(f)\). We now distinguish two cases. In the first case \(\lim_{x \to a} f(x) = l = 0\). In this case, given \(\varepsilon > 0\), there is \(\delta > 0\) such that
\[
|f(x)| - |l| = |f(x)| = |f(x) - l| < \varepsilon
\]
if \(0 < |x - a| < \delta\) and \(x \in \text{dom}(f)\), since \(\lim_{x \to a} f(x) = l\). In the second case we can assume without loss of generality that \(l > 0\). Then there is \(\delta_0 > 0\), such that \(f(x) > l/2\) if \(0 < |x - a| < \delta_0\). In particular, given any \(\varepsilon > 0\), we can take \(\delta_1 > 0\) such that \(0 < |x - a| < \delta_1\) implies \(|f(x) - l| < \varepsilon\). If we take \(\delta = \min(\delta_0, \delta_1)\), we get that
\[
|f(x)| - |l| = |f(x) - l| < \varepsilon
\]
if \(0 < |x - a| < \delta\).

(b) One easily checks that \(\max(f, g) = \frac{1}{2}(f + g + |f - g|)\) and \(\min(f, g) = \frac{1}{2}(f + g - |f - g|)\). Therefore, the assertion follows from part (a) and the additivity of limits proved in class. Note, that we have to assume(!) here, that the functions \(\max(f, g)\) and \(\min(f, g)\) are defined for points arbitrarily close to \(a\).

Problem 2 (a) If \(\lim_{x \to 0} 1/x = l \in \mathbb{R}\), take \(\delta > 0\), such that \(0 < |x| < \delta\) implies \(|1/x - l| < 1\). for such \(x\) we get from the triangle inequality that
\[
|1/x| \leq |1/x - l| + |l| \leq |l| + 1.
\]
If we take \(x\) such that \(0 < |x| < \min(\delta, 1/(|l| + 2))\), we find that
\[
|l| + 2 \leq |l| + 1
\]
which is a contradiction.

(b) Either use the same argument or observe that if \(\lim_{x \to 1} 1/(x - 1)\) would exist, also \(\lim_{y \to 0} 1/y\) would exist.

Problem 3 (a) Clearly, there are points arbitrarily close but not equal to 0 at which \(g(x) \sin(1/x)\) is defined. Given \(\varepsilon > 0\), take \(\delta > 0\) such that \(|g(x)| < \varepsilon\) if \(0 < |x| < \delta\). Then also
\[
|g(x) \sin(1/x)| \leq |g(x)| < \varepsilon
\]
if \(0 < |x| < \delta\).

(b) Assume (!) that \(gh\) is defined for points arbitrarily close but not equal to 0. Given \(\varepsilon > 0\), take \(\delta > 0\) such that \(0 < |x| < \delta\) implies \(|g(x)| < \varepsilon/M\). Then also
\[
|g(x)h(x)| \leq M|g(x)| < \varepsilon
\]
if \(0 < |x| < \delta\).

Problem 4 (a) Consider the function \(f : \mathbb{R} \to \mathbb{R}\) defined by
\[
f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}
\]
We know that \( \lim_{x \to 0} f(x) \) does not exist. But if we take \( \epsilon = 2 \) then
\[
|f(x) - 0| < \epsilon
\]
for any \( x \).

(b) Consider the function \( g: [0, \infty) \to \mathbb{R} \) defined by
\[
f(x) = \begin{cases} 
1 & \text{if } x = 1/n \text{ for some } n \in \mathbb{N} \\
1 & \text{if } x = 0 \\
x & \text{else.}
\end{cases}
\]
We have seen in various similar examples that \( \lim_{x \to 0} g(x) \) does not exist. On the other hand, given \( \epsilon > 0 \), we can take \( \delta = \min(1, \epsilon) \) and if \( |g(x)| < \epsilon \) we need to have \( 0 < |x| < \delta \).

**Problem 5** (a) Fix \( K > 0 \). Since \( \lim_{x \to 3} (x - 3)^2 = 0 \), there is \( \delta > 0 \) such that if \( 0 < |x - 3| < \delta \), then \( |(x - 3)^2| < 1/K \), which implies \( 1/(x - 3)^2 > K \). (b) Let’s rewrite things a tiny little bit. We assume that there is a constant \( c > 0 \) such that \( f(x) > c \) for any \( x \) and that \( \lim_{x \to a} g(x) = 0 \). Given \( K > 0 \), take \( \delta > 0 \) such that \( |g(x)| < c/K \) if \( 0 < |x - a| < \delta \). Then, for such \( x \), we also have that
\[
\frac{f(x)}{|g(x)|} > \frac{c}{c/K} = K.
\]
Since \( K > 0 \) was arbitrary, we get that \( \lim_{x \to a} (f(x)/|g(x)|) = +\infty \).

**Problem 6** (a) Given \( \epsilon > 0 \), take \( \delta = \epsilon \), then if \( |x| < \delta \), we have that
\[
|f(x) - f(0)| = |f(x)| \leq |x| < \epsilon.
\]
(b) Take
\[
f(x) = \begin{cases} 
x & \text{if } x \in \mathbb{Q} \\
0 & \text{else.}
\end{cases}
\]
The same argument that we used for
\[
g(x) = \begin{cases} 
1 & \text{if } x \in \mathbb{Q} \\
0 & \text{else.}
\end{cases}
\]
shows that \( f \) is not continuous at any \( x \neq 0 \). (The argument only used that there are jumps of definite size arbitrarily close to \( x \). As long as \( x \neq 0 \) this is also true for \( f \).) By part (a), \( f \) is continuous at 0.
(c) Given \( \epsilon > 0 \), take \( \delta > 0 \) such that \( |x| < \delta \) implies that \( |g(x)| < \epsilon \). For such \( x \) we also have that
\[
|f(x) - f(0)| = |f(x)| \leq |g(x)| < \epsilon.
\]
Problem 7 (a) Consider \( f : \mathbb{R} \to \mathbb{R} \) given by

\[
f(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
1 & \text{if } x \geq 1 \\
\frac{1}{n+1} & \text{if } x \in \left( \frac{1}{n}+1, \frac{1}{n} \right] \text{ for } n \in \mathbb{N}.
\end{cases}
\]

Then \( f \) is constant on each of the open (!!!) (why is this important?) intervals \((\infty, 0), (1), \left( \frac{1}{n+1}, \frac{1}{n} \right), \text{ and } (1, \infty)\), hence continuous. By Problem 6 part (a), \( f \) is continuous at 0. Moreover, the argument we have used multiple times in the presence of a jump of definite size shows that \( f \) is not continuous at any of the points \( 1/n, n \in \mathbb{N} \). (b) Simply take \( f \) from the preceding part and make it discontinuous at 0 by taking any value different from 0 at 0.

Problem 8 Let’s define \( g : \mathbb{R} \to \mathbb{R} \) as follows

\[
g(x) = \begin{cases} 
f(x) & \text{if } x \in [a, b] \\
f(b) & \text{if } x \in (b, \infty) \\
f(a) & \text{if } x \in (-\infty, a).
\end{cases}
\]

Note that \( g \) is continuous on each of the open (again, why is this important?) intervals \((a, b), (b, \infty), \text{ and } (-\infty, a)\) since it agrees with the continuous function \( f \) or is constant, respectively. So we only need to check that \( g \) is continuous at \( a \) and \( b \). Given \( \varepsilon > 0 \), there is \( \delta > 0 \) such that if \( 0 \leq x - a < \delta \), then

\[
|g(x) - g(a)| = |f(x) - f(a)| < \varepsilon.
\]

Moreover, if \( 0 \leq a - x < \delta \), then

\[
|g(x) - g(a)| = |f(a) - f(a)| = 0 < \varepsilon.
\]

It follows that \( g \) is continuous at \( a \). The exact same argument shows that \( g \) is also continuous at \( b \).

(b) Take \( f : (0, 1) \to \mathbb{R} \) given by \( x \mapsto 1/x \). We have seen that \( \lim_{x \to 0} f(x) \) does not exist. In particular, there does not exist any continuous \( g : [0, 1] \to \mathbb{R} \) such that \( f = g \) in \((0, 1)\).