Problem 1 (a) Given two sets $A, B$, let $(a, b), (c, d) \in A \times B$. Prove from the definition given in class that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$ assuming the fact that a set cannot contain itself.

(b) Given two sets $A, B$ and $a \in A$ and $b \in B$, define 

$$[a, b] := \{a, \{b\}\}.$$ 

Give an example of two sets $A, B$ and elements $(a, b), (c, d) \in A \times B$, such that $[a, b] = [c, d]$, but $(a, b) \neq (c, d)$.

Problem 2 Let $K = \{0, 1\}$ and define $+: K \times K \to K$ by 

$0 + 0 = 0, 0 + 1 = 1 + 0 = 1, 1 + 1 = 0$ and $\cdot: K \times K \to K$ by 

$0 \cdot 0 = 0, 0 \cdot 1 = 1 \cdot 0 = 0, 1 \cdot 1 = 1$. Show that $(K, +, \cdot)$ is a field. Can you find a field $(K, +, \cdot)$ such that $1 + 1 + 1 = 0$?

Problem 3 Spivak 1.5

Problem 4 Spivak 1.7

Problem 5 Spivak 1.8

Problem 6 Spivak 2.1

Problem 7 Spivak 2.3 (those parts not done in class)

Problem 8 Spivak 2.5

Problem 9 Spivak 2.19

Problem 10 For any $x > 0$ and $n \in \mathbb{N}$ with $n \geq 2$, we have 

$$(1 + x)^n \geq \frac{n^2 x^2}{4}.$$ 

Problem 11* (extra credit) (a) Let $n \in \mathbb{N}$ and $p_1, \ldots, p_l$ the primes in 

$\{1, \ldots, 2n\}$ ordered by size. Choose $m$, such that $p_{l-m+1} \geq n + 1$ but 

$p_{l-m} < n + 1$. Prove that 

$$\binom{2n}{n} \geq \prod_{j=l-m+1}^{l} p_j.$$ 

Hint: It is helpful to consider the prime decompositions of $\binom{2n}{n}$ and $n!$ and to recall that $\binom{2n}{n} \in \mathbb{N}$.

(b) Use part (a) to show that 

$n^m \leq 2^{2n}$. 