A SHORT PROOF THAT ADDITIVE, MEASURABLE FUNCTIONS ARE LINEAR

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For notation, we use $\lambda$ to mean Lebesgue measure on $\mathbb{R}$.

**Lemma 0.1.** If $A, B$ are sets of positive measure, then there exists $q \in \mathbb{Q}$ such that $\lambda((A + q) \cap B) > 0$.

**Proof.** By the Lebesgue density theorem we get $\Delta > 0$, $a \in A$ and $b \in B$ such that $\lambda(A \cap [a - \delta, a + \delta])$, $\lambda(B \cap [b - \delta, b + \delta]) \geq \delta/2$ for all $\delta \leq \Delta$. If we assume that $\lambda((A + q) \cap B) = 0$ for all $q \in \mathbb{Q}$ then for any $q \in \mathbb{Q}$ such that $a + q$ is close to $b$, we get

$$2\Delta \geq \lambda((A + q) \cap [a + q - \Delta, a + q + \Delta]) \cup (B \cap [a + q - \Delta, a + q + \Delta])$$

$$\geq \lambda(A \cap [a - \Delta, a + \Delta]) + \lambda(B \cap [b - (\Delta - |b - (a + q)|), b + (\Delta - |b - (a + q)|)])$$

$$\geq (3/2)\Delta + (3/2)\Delta - (3/2)|b - (a + q)|$$

Since we can choose $q$ to make the last term as small as we please, we get a contradiction. \qed

**Lemma 0.2.** If $A$ is a set with $\lambda(A) > 0$ which has the property that $A +q = A$ for every $q \in \mathbb{Q}$ then $\lambda(A^c) = 0$.

**Proof.** Notice that for all $q$, $(A + q) \cap A^c = \emptyset$. Hence, $\lambda(A^c) > 0$ would contradict the lemma above. \qed

**Theorem 0.3.** If $f : \mathbb{R} \to \mathbb{R}$ is additive and measurable then $f$ is linear.

**Proof.** Let $g(x) = f(x) - f(1)x$ and we are finished if we show that $g$ is 0. Notice that $g$ is linear and measurable as well. We first show that $g$ takes the value 0 on the rational numbers. Notice that $g(1) = 0$. Now, take a rational number $\frac{p}{q}$ and we get that:

$$0 = pg(1) = g(1) + \cdots + g(1) = g(p) = g(p/q) + \cdots + g(p/q) = qg(p/q)$$

Hence $g(p/q) = 0$. An important consequence of this is that $g(x + r) = g(x)$ whenever $r \in \mathbb{Q}$.

Now we show that $g$ is essentially bounded. Suppose this isn’t true. Then for every $M$, the set $A_M = \{x : |g(x)| \geq M\}$ has positive measure. Since, for any $q \in \mathbb{Q}$, $g(x + q) = g(x)$, then $A_M = A_M + q$. By the lemma above, $\lambda(A_M^c) = 0$. Notice that this holds for all $M$. However, since $g : \mathbb{R} \to \mathbb{R}$ is necessarily finite, then $\mathbb{R} = \bigcup_n A_n^c$. Hence $0 = \lim \lambda(A_n^c) = \lambda(\mathbb{R}) = \infty$. This is clearly a contradiction.

Now, we know that $g$ is integrable on compact sets and that it is periodic. Hence, for any $y \in \mathbb{R}$ we have

$$\int_0^1 g(x)dx = \int_0^{1-y} g(x+y)dx = \int_0^1 g(x+y)dx = \int_0^1 g(x)dx + \int_0^1 g(y)dx = g(y) + \int_0^1 g(x)dx$$

This clearly implies $g(y) = 0$ for all $y$, finishing the proof. \qed