Math 27300, Homework 5
Due Tuesday, February 12

Exercises: 6.11, 6.12 (a)-(e), 6.13

Let $T = \mathbb{R}/2\pi \mathbb{Z}$. Roughly, $T = [0, 2\pi]$ with the ends identified (that is, $0 \sim 2\pi$).

**Exercise 1.** Let

$$A = \begin{pmatrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{pmatrix}$$

and $X' = AX$. We write $X(t) = (x_1(t), y_1(t), x_2(t), y_2(t))$. Suppose that $x_1(0)^2 + y_1(0)^2 = x_2(0)^2 + y_2(0)^2 = 1$ and that $\omega_2/\omega_1 \in \mathbb{R} \setminus \mathbb{Q}$.

(i) Without appealing to the general solution of the system, show that for $i = 1, 2$, $x_i(t)^2 + y_i(t)^2 = 1$ for all $t$.

(ii) Since $(x_i(t), y_i(t)) \in S^1$ for all $t$, we may define $\theta_i : \mathbb{R} \to \mathbb{T}$ be such that $(x_i(t), y_i(t)) = (\sin(\theta_i(t)), \cos(\theta_i(t)))$ for all $t$. Derive an ODE for $\theta_i$ and use it to show that $\theta_i(t) = \theta_i(0) + \omega_i t$ for $i = 1, 2$.

(iii) For any $z_0$, recall that the Poincaré map $p : \mathbb{T} \to \mathbb{T}$ is given by:

$$p(z_0) = z_0 + \frac{2\pi \omega_2}{\omega_1}.$$

The interpretation is that $p(z_0)$ is the value of $\theta_2(t_0)$ where $t_0$ is the first positive time that $\theta_1(t_0) = 0$ given that $\theta_1(0) = 0$ and $\theta_2(0) = z_0$. Let $z_k = p(z_{k-1})$ for all $k \in \mathbb{N}$. For every $\epsilon$, prove that there exists $k, \ell \in \mathbb{N}$ such that $|z_k - z_{k+\ell}| < \epsilon$. [Hint: use the pigeonhole principle]

(iv) Let $\delta = |z_k - z_{k+\ell}| < \epsilon$ and assume that $z_k + \ell = z_k + \delta$. Fix any $y \in \mathbb{T}$ and show that there exists $m \in \mathbb{N}$ such that $|z_{k+m\ell} - y| \leq \delta$. Deduce that the “orbit” of $z_0$, $\mathcal{O}_{z_0} = \{z_0, z_1, z_2, \ldots \}$, is dense in $\mathbb{T}$.

(v) Prove that $\{(\theta_1(t), \theta_2(t)) : t \in \mathbb{R}\}$ is dense in $\mathbb{T} \times \mathbb{T}$.

Note: this exercise establishes that the trajectory of $X(t)$ is dense in $S^1 \times S^1$. This is the quintessential example of “ergodic” dynamics – in a very weak sense, it “mixes” everything up.

**Exercise 2.** In this exercise, we develop the exponential of a matrix. Let $A \in \mathbb{R}^{n \times n}$.

(i) Show that there exists $C_A > 0$ such that, for any $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$, $\|A^k x\| \leq C_A^k \|x\|$.

(ii) For each $m \in \mathbb{N}$, define $B_m \in \mathbb{R}^{n \times n}$ by $B_m = \sum_{k=0}^{m} \frac{1}{k!} A^k$. Show that there exists $B \in \mathbb{R}^{n \times n}$ such that, for all $x \in \mathbb{R}^n$, $Bx = \lim_{m \to \infty} B_m x$. [Hint: do not try to write down the matrix $B$. Instead, first show that $B_m x$ is a Cauchy sequence in $\mathbb{R}^n$, then define $B : \mathbb{R}^n \to \mathbb{R}^n$ as the function such that $Bx = \lim_{m \to \infty} B_m x$, and lastly, verify that $B$ is a linear transformation.]
(iii) The matrix $B$ in part (ii) can be denoted $\exp\{A\}$ or $e^A$. Suppose that $T \in \mathbb{R}^{n \times n}$ is invertible. Then
\[ T e^A T^{-1} = e^{T A T^{-1}}. \]

(iv) If $M \in \mathbb{R}^{n \times n}$ and $MA = AM$, show that $e^A e^M = e^{A+M}$.

(v) Show that $e^{-A} = (e^A)^{-1}$.

(vi) For every $t \in \mathbb{R}$, let $B_t = \exp\{tA\}$. Prove that $B_{t+s} = B_t B_s$ and that, for all $x \in \mathbb{R}$,
\[ \lim_{t \to 0} B_t x = x. \]

(vii) Fix any $x \in \mathbb{R}^n$ and prove that the function $X : \mathbb{R} \to \mathbb{R}^n$ defined by $X(t) = B_t x$ is continuously differentiable. Show that $X'(t) = AX(t)$ for all $t \in \mathbb{R}$. Finally, show that if $Y'(t) = AY(t)$ for all $t \in \mathbb{R}$ and $Y(0) = x$, then $Y(t) = X(t)$ for all $t \in \mathbb{R}$.

**Exercise 3.** Find the solution to $X' = AX + F(t)$ where $X(0) = (-1, 1)$,
\[ A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad \text{and} \quad F(t) = \begin{pmatrix} e^{-2t} \\ e^{-t} \end{pmatrix} \quad \text{for all } t \in \mathbb{R}. \]