Math 27200, Homework 4  
Due Tuesday, April 30

**Exercise 1** (Royden-Fitzpatrick 13.30). Let $X$ be a Banach space and $P \in \mathcal{L}(X, X)$ be a projection. Show that $P$ is open.

**Exercise 2** (Royden-Fitzpatrick 13.31). Let $X$ and $Y$ be Banach spaces and $T \in \mathcal{L}(X, Y)$ . Show that $T$ is open if the image under $T$ of the open unit ball in $X$ is dense in a neighborhood of the origin in $Y$.

**Exercise 3.** Suppose that $X$ is a Banach space and $T : X \to \mathbb{R}$ is discontinuous. Show that $T^{-1} (\{0\})$ is dense in $X$.

**Exercise 4** (Royden-Fitzpatrick 13.34). Let $X$ be a banach space, $T \in \mathcal{L}(X, X)$ be open, and $X_0$ be a closed subspace of $X$. The restriction $T_0$ of $T$ to $X_0$ is continuous. Is $T_0$ necessarily open?

**Exercise 5** (Royden-Fitzpatrick 13.35). Let $V$ be a linear subspace of a linear space $X$. Argue as follows that $V$ has a linear complement in $X$.

(i) If $\dim X < \infty$, let $\{e_i\}_{i=1}^n$ be a basis for $V$. Extend this basis for $V$ to a basis $\{e_i\}_{i=1}^{n+k}$ for $X$. Then define $W = \text{span}\{e_{n+1}, \ldots, e_{n+k}\}$.

(ii) If $\dim X = \infty$, apply Zorn’s Lemma to the collection $\mathcal{F}$ of all subspaces of $Z$ of $X$ for which $V \cap Z = \{0\}$, ordered by set inclusion.

**Exercise 6** (Royden-Fitzpatrick 13.37). Let $Y$ be a normed linear space. Show that $Y$ is a Banach space if and only if there is a Banach space $X$ and a continuous, linear, open mapping of $X$ onto $Y$.

**Exercise 7** (Royden-Fitzpatrick 13.39). Let $\{f_n\}$ be a sequence in $L^\infty[a, b]$ for $a < b$. Suppose that for each $g \in L^1[a, b]$, $\lim_{n \to \infty} \int_a^b g(x)f_n(x)dx$ exists. Show that there is $f \in L^\infty[a, b]$ such that $\int_a^b g f = \lim_{n \to \infty} \int_a^b g f_n$ for all $g \in L^1[a, b]$.

**Exercise 8.** Suppose that $X$ and $Z$ are a Banach spaces and $Y \subset X$ is a dense subspace of $X$. If $T \in \mathcal{L}(Y, Z)$, then there is a unique extension $\tilde{T} \in \mathcal{L}(X, Z)$ such that $\tilde{T}|_Y = T$.

**Exercise 9.** Define the Fourier transform $\mathcal{F} : L^1(\mathbb{R}; \mathbb{C}) \to L^\infty(\mathbb{R}; \mathbb{C})$ by

$$\mathcal{F} f(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-ix\xi} f(x)dx.$$  

1. Show that $\mathcal{F} \in \mathcal{L}(L^1, L^\infty)$.

2. For any $\sigma > 0$, let

$$G_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma}}.$$  

Show that $\mathcal{F}G_\sigma = G_{\sigma^{-1}}$.

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*Here we use $L^\infty(\mathbb{R}; \mathbb{C})$ to denote the space of almost everywhere bounded functions whose domain is $\mathbb{R}$ and whose codomain is $\mathbb{R}$.***
3. Define $T_\sigma : L^1 \cap L^2 \rightarrow L^1 \cap L^2$ by

$$(T_\sigma f)(x) = G(x) * f(x) = \int_{\mathbb{R}} G(x-y)f(y)dy.$$ 

Show that $T_\sigma \rightarrow id$. Note: $\|\cdot\|_{L^1 \cap L^2} := \|\cdot\|_{L^1} + \|\cdot\|_{L^2}$.

4. Show that if $f, g \in L^1 \cap L^2$, then $\mathcal{F}(G_\sigma * f) = (\mathcal{F}G_\sigma)(\mathcal{F}f)$ and

$$\int_{\mathbb{R}} f(x)g(x)dx = \int_{\mathbb{R}} \mathcal{F}f(\xi)\mathcal{F}g(\xi)d\xi.$$ [Hint: Part (iii) is useful here!]

5. Show that $\mathcal{F}$ extends to an isometry in $\mathcal{L}(L^2, L^2)$.

**Definition 10.** Let $X$ and $Y$ be Banach spaces. We say that $T \in \mathcal{L}(X, Y)$ is a **compact operator** if $T(B_1(0))$ is a compact set.

**Exercise 11.**

1. Suppose that $X$ is a Banach space. Show that id : $X \rightarrow X$ is compact if and only if $X$ is finite dimensional.

2. Give an example of an infinite dimensional Banach space and an injective map on it that is compact.