Exercise 1 (Royden-Fitzpatrick 10.4). Let $S$ be a countable set and $\{f_n\}$ be a sequence of real-valued functions on $S$ that is pointwise bounded on $S$. Show that there is a subsequence of $\{f_n\}$ that converges pointwise on $S$ to a real-valued function.

Exercise 2 (Royden-Fitzpatrick 13.4). For a normed linear space $X$, show that the function $f : X \to \mathbb{R}$ such that $f(x) = \|x\|$ for all $x \in X$ is continuous.

Exercise 3 (Royden-Fitzpatrick 13.11). For $X$ and $Y$ normed linear spaces and $T \in \mathcal{L}(X,Y)$, show that 

$$
\|T\| = \sup \{\|T(u)\| : u \in X, \|u\| \leq 1\}.
$$

Exercise 4 (Royden-Fitzpatrick 13.12). For $X$ a normed linear space and $T,S \in \mathcal{L}(X,X)$, show that the composition $S \circ T$ also belongs to $\mathcal{L}(X,X)$ and

$$
\|S \circ T\| \leq \|S\| \cdot \|T\|.
$$

Exercise 5. Fix $1 \leq p \leq \infty$. Show that $\ell^p(\mathbb{N})$ is a complete metric space.

Exercise 6. (i) Suppose that $p,q \in [1,\infty]$. When is $\ell^p(\mathbb{N}) \subset \ell^q(\mathbb{N})$? Prove or give counterexamples for all cases.

(ii) Suppose that $x \in \ell^{p_0}(\mathbb{N})$ for some $p_0 \geq 1$. Show that 

$$
\lim_{p \to \infty} \|x\|_{\ell^p} = \|x\|_{\ell^\infty}.
$$

Exercise 7. Let $c = \{\{x_n\} \in \ell^\infty : \text{there exists } N \in \mathbb{N} \text{ such that if } n \geq N, x_n = 0\}$.

(i) Show that $c$ is a normed linear space but is not complete. Here we endow $c$ with the $\ell^\infty$ norm.

(ii) Find the completion of $c$.

Exercise 8. Consider $C^1([0,1]) = \{f : [0,1] \to \mathbb{R} : f \text{ is differentiable and } f' \in C([0,1])\}$ as a metric subspace of $L^2([0,1])$.

(i) Show that $(C^1([0,1]), \|\cdot\|_{H^1})$ is a normed linear space but is not complete. Here let $\|f\|_{H^1} = \|f\|_{L^2} + \|f'\|_{L^2}$.

(ii) Let $H^1([0,1])$ be the completion of $C^1([0,1])$. Give an explicit example of a function $f \in H^1([0,1]) \setminus C^1([0,1])$.

(iii) Show that if $f \in H^1([0,1])$, then $f \in C^{0,1/2}([0,1])$, where

$$
C^{0,1/2}([0,1]) := \{f \in C([0,1]) : \exists K \text{ s.t. } |f(x) - f(y)| \leq K|x-y|^{1/2} \forall x,y \in [0,1]\}.
$$

[Hint: use Holder’s inequality and the Fundamental Theorem of Calculus.]

Exercise 9. A metric space $(X, \rho)$ is called separable if there exists a countable, dense subset of $X$. Show that any compact metric space is separable. [Hint: for any $n$, use compactness to find $x_{1,n}, \ldots, x_{k_n,n}$ such that $X \subset \bigcup_{i=1}^{k_n} B_{1/n}(x_{i,n})$]
Exercise 10. Recall that if \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) are normed linear spaces and \(T : X \to Y\) is a linear transformation, then the norm of \(T\) is given by
\[
\|T\|_{\mathcal{L}(X,Y)} := \sup_{0 \neq x \in X} \frac{\|Tx\|_X}{\|x\|_Y}.
\]
Find the norm of each of the following linear transformations.

(i) \(A : L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d), Af = 57f\)

(ii) \(S : c \to c, S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)\)

(iii) \(P : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), Pf = \lambda_1f_1 + \lambda_2f_2 + \ldots, \text{ where}^1 f_1, f_2, \ldots \in L^2(\mathbb{R}^d), \int f_if_jdx = \delta_{ij}, \text{ and} \lambda_i = \int f_i^2dx.\)

Exercise 11 (Weak convergence). Let \(f_n : \mathbb{R} \to \mathbb{R}\) be defined by \(f_n(x) = 1\) if \(x \in [n, n+1]\) and 0 if \(x \notin [n, n+1]\). Hint: for this exercise, it will be useful to show that if \(1 \leq p < \infty, f \in L^p(\mathbb{R})\), then, for all \(\epsilon > 0\), there exists \(K_\epsilon \subset \mathbb{R}\), which is compact, such that \(\|1_{K_\epsilon}f\|_{L^p} \leq \epsilon.\)

(i) Fix any \(p \in (1, \infty)\). Show that \(f_n \rightharpoonup 0 \text{ in } L^p(\mathbb{R}).\)

(ii) Show that \(f_n \not\rightharpoonup f\) in \(L^1(\mathbb{R})\) for any \(f \in L^1(\mathbb{R}).\) [Hint: suppose that \(f_n \rightharpoonup f\) for some \(f \neq 0.\) Use the hint above to get a contradiction. Then use the observation from class to show that \(f_n \not\rightharpoonup 0.\)]

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^1Recall that \(\delta_{ij}\) is 1 if \(i = j\) and 0 otherwise.