
Exercise 1. Fix any $p \in (1, \infty)$. Let $(X_p, \rho_p)$ be as defined in lecture, and let $q = p/(p - 1)$ so that $p^{-1} + q^{-1} = 1$. Here we use the notation that $\|f\|_p = \rho_p(f, 0) = \left( \int |f|^p dx \right)^{1/p}$.

(i) (Young’s inequality) Show that, for any real numbers $a, b \geq 0$, $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$. [Hint: it may be useful to use calculus to establish that $x^p/p + 1 - x$ is non-negative for any $x \geq 0$]

(ii) (Hölder’s inequality) Show that $\int fg dx \leq \|f\|_p \|g\|_q$ for any $f \in X_p$ and $g \in X_q$. [Hint: it may be useful to use that Young’s inequality on $(\alpha f)(g/\alpha)$ for a well-chosen $\alpha > 0$]

(iii) (Minkowski’s inequality) Show that $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ for all $f, g \in X_p$. [Hint: use (ii)]

(iv) Show that $(X_p, \rho_p)$ is a metric space.

(v) Show that $(X_p, \rho_p)$ is not complete.

Exercise 2. Let $C([0,1]) = \{f : [0,1] \to \mathbb{R} | f$ is continuous$\}$. Show that $(C([0,1]), \rho_\infty)$ is a complete metric space.