Math 16110
Fall 2017
Final Exam
December 7, 2017
Time Limit: 2 Hours

Name (Print): ____________________

This exam contains 11 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit** unless otherwise indicated. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit (unless otherwise indicated); an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **Do not use theorems from the notes that appear later than** the material from the question. When there is some ambiguity, ask the proctor!
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	15	
4	15	
5	10	
6	10	
7	20	
8	10	
Total:	100	

1. (10 points) Fix a set C satisfying Axioms 1, 2, and 3. Prove that if $x, y \in C$ and if $x < z$ for all $z > y$, then $x \leq y$.

2. (a) (3 points) Finish the definition: A finite set A has cardinality n, for some $n \in \mathbb{N}$, if...

(b) (3 points) Prove that $|A| = 7$ where $A = \{1, 2, 10, 5, 7, 20^{17}, 4\}$.

(c) (4 points) Finish the definition: A set A is countable if...

3. Let $C = \mathbb{Z} \times \mathbb{N}$ with the ordering $(a, b) < (c, d)$ if either $a < c$ or both $a = c$ and $b < d$.

   (a) (10 points) Find all limit points of $\{(z, k) \in C : z > 4, k > z\}$.

   (b) (5 points) Is C closed? Is C open?

4. (15 points) Prove the pigeon-hole principle: If $m, n \in \mathbb{N}$ and $m > n$, then there does not exist a bijection $f : [m] \to [n]$. (Hint: for any fixed $k \in \mathbb{N}$, prove, by induction on n, that there does not exist an injection $f : [n + k] \to [n]$.)

5. Suppose that C is a continuum

   (a) (6 points) Prove that if $p, q \in C$ and $p \neq q$ then there exists regions R_p and R_q such that $p \in R_p$, $q \in R_q$, and $R_p \cap R_q = \emptyset$.

   (b) (4 points) Prove that if $n \in \mathbb{N}$, $p_1, \ldots, p_n \in C$, and $p_i = p_j$ if and only if $i = j$ then there exists regions $R_1, \ldots, R_n$ such that $p_i \in R_i$ for all i and $R_i \cap R_j = \emptyset$ for all $i \neq j$.

6. Answer each question as True or False. No partial credit awarded.

(a) (2 points) $\mathbb{N}$ is a continuum.

(b) (2 points) If $A_1, A_2, \cdots \subset C$ are closed sets, then $\cup_{i=1}^{\infty} A_i$ is closed.

(c) (2 points) The set $X = \{(x, y) \in \mathbb{N} \times \mathbb{Z} : x - 27 = y\}$ is a function with domain $\mathbb{N}$ and codomain $\mathbb{Z}$.

(d) (2 points) $\wp(\mathbb{Z})$ is countable.

(e) (2 points) Suppose that $\underline{ab} \subset C$ is a region. Then $\underline{ab}$ is a continuum.

7. (10 points) Define an equivalence relation $\sim$ on $\mathbb{Z} \times \mathbb{Z}_{>0}$, where $\mathbb{Z}_{>0} = \{z \in \mathbb{Z} : z > 0\}$, by $(a,b) \sim (c,d)$ if $ad = bc$. Define $\mathbb{Q}' = \{[(a,b)] : (a,b) \in \mathbb{Z} \times \mathbb{Z}_{>0}\}$, $[(\alpha,\beta)] +_{\mathbb{Q}} [(\gamma,\delta)] = [(\alpha\delta + \beta\gamma, \beta\delta)]$, and $[(\alpha,\beta)] <_{\mathbb{Q}} [(\gamma,\delta)]$ if $\alpha\delta < \beta\gamma$. You may assume that $+_{\mathbb{Q}}$ is well-defined.

   (a) (5 points) Show that $<_{\mathbb{Q}}$ is well-defined.

   (b) (5 points) Suppose that $[(a,b)] \geq [(0,1)]$ and $[(\alpha,\beta)] >_{\mathbb{Q}} [(\gamma,\delta)]$. Show that $[(a,b)] +_{\mathbb{Q}} [(\alpha,\beta)] >_{\mathbb{Q}} [(a,b)] +_{\mathbb{Q}} [(\gamma,\delta)]$.

8. Give an example of an object with the given properties. No partial credit awarded.

   (a) (2 points) A continuum C where every set is closed.

   (b) (2 points) A subset $X \subset \mathbb{Z} \times \mathbb{Z}$ that is *not* a function with domain $\mathbb{Z}$ and codomain $\mathbb{Z}$.

   (c) (2 points) A set A and a continuum C such that $A \subset C$ and A is neither open nor closed.

   (d) (2 points) A countable set that is not $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$.

   (e) (2 points) Given $\underline{ab} \in \mathbb{Q}$, an open set $V \subsetneq \underline{ab}$.

(Space for scratch work)

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