

Math 161, Homework 3  
Due in class Tuesday, October 17

**Definition 1.** Consider any two sets  $A$  and  $B$ . We say that  $A$  has the *same cardinality* as  $B$  if there exists a bijection  $f : A \rightarrow B$ . We say that  $A$  has *smaller cardinality* than  $B$  if there exists an injection  $f : A \rightarrow B$ .

**Exercise 2.** Suppose that  $A \subset B$ , prove that  $A$  has smaller cardinality than  $B$ .

**Exercise 3.** Fix any two sets  $A$  and  $B$ .

- (a) Suppose that  $A$  has the same cardinality as  $B$ . Prove that  $B$  has the same cardinality as  $A$ . (Hence, we may say that  $A$  and  $B$  have the same cardinality.)
- (b) Suppose that  $A$  and  $B$  have the same cardinality. Show that  $\wp(A)$  and  $\wp(B)$  have the same cardinality.

**Exercise 4.** Suppose that  $A, B \subset \mathbb{N}$  and that there exists  $f : A \rightarrow B$  which is surjective. Find a property  $P$  which will make the following claim true.

Claim: If  $B$  is  $P$  and there is a surjection  $f : A \rightarrow B$ , then  $A$  and  $B$  have the same cardinality.

Prove this with your choice of  $P$  and give a counter-example where the claim fails when  $P$  is not satisfied.

**Definition 5.** Given two sets  $A$  and  $B$ , we say that  $A \subsetneq B$  if  $A \subset B$  and  $A \neq B$ .

**Exercise 6.** Using Definition 2.1, for each set  $X$  and subset  $<_X \subset X \times X$ , determine if  $<_X$  is an ordering:

- (a) Let  $X = \{\{x \in \mathbb{N} \mid x \leq n\} \in \wp(\mathbb{N}) \mid n \in \mathbb{N}\}$  and  $<_X = \{(A, B) \in X \times X \mid A \subsetneq B\}$ .
- (b) Let  $X = \{f \subset \mathbb{N} \times \mathbb{N} \mid f \text{ is a function}\}$  and  $<_X = \{(f, g) \in X \times X \mid f(n) < g(n) \text{ for all } n \in \mathbb{N}\}$ .
- (c) Let  $X = \{f \subset \mathbb{N} \times \mathbb{N} \mid f \text{ is a function}\}$  and  $<_X = \{(f, g) \in X \times X \mid f(n) \leq g(n) \text{ for all } n \in \mathbb{N} \text{ and there exists } n \in \mathbb{N} \text{ such that } f(n) < g(n)\}$ .

(For the purposes of this exercise, “ $<$ ” means the usual ordering on  $\mathbb{N}$ , i.e. if  $m, n \in \mathbb{N}$  then  $m < n$  if and only if  $n = m + k$  for some  $k \in \mathbb{N}$ .)

**Exercise 7.** Let  $A$  be any set. Show that there is no bijection between  $A$  and its power set  $\wp(A)$ . (Hint: If  $f : A \rightarrow \wp(A)$  is any function, think about the set  $B = \{a \in A \mid a \notin f(a)\} \subset A$ .)

**Exercise 8.** Show that there is no injective map from  $\wp(\mathbb{N})$  to  $\mathbb{N}$ .