GENERAL GALOIS DEFORMATIONS

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Abstract. These are notes I prepared for a talk at the Modularity Lifting seminar. I claim no originality to the contents of these notes. Nor do I claim that they are without errors, nor readable.

Throughout this talk, let $p$ be a prime, $F$ a finite field of characteristic $p$, $G$ a profinite group, a representation $\bar{\rho} : G \to GL_n(F)$ or $GL(V_F)$ (depending on whether or not we fix a basis). Let $\Lambda = W(F)$ (can also be $O_K$ for some finite extension $K/\mathbb{Q}_p$).

1. Deformation Functor

Definition 1.1. Let $\hat{\mathfrak{A}}_\Lambda$ be the category of complete Noetherian local $\Lambda$-algebras with residue field $F$ (with local morphisms).

Let $\mathfrak{A}_\Lambda$ be the full subcategory of finite local Artinian $\Lambda$-algebras.

Remark 1.2. Notice that for $A \in \mathfrak{A}_\Lambda$, $\lim\limits_{\leftarrow} A/m_i^i A$ is an object of $\hat{\mathfrak{A}}_\Lambda$.

Define the deformation functor $\text{Def}(\bar{\rho}) : \hat{\mathfrak{A}}_\Lambda \to \text{Sets}$ by

$$\text{Def}(\bar{\rho})(A) = \{(\rho, M, i)\} / \sim$$

where

- $M$ is a free $A$-module of rank $n$
- $\rho : G \to GL_A(M)$ is a continuous representation
- $i : \rho \otimes_A F \cong \bar{\rho}$ is an isomorphism

and the equivalence is isomorphism of representations compatible with $i$.

Fix a basis $\beta_F$ of $V_F$, and define the framed deformation functor $\text{Def}^\Box(\bar{\rho}) : \hat{\mathfrak{A}}_\Lambda \to \text{Sets}$ by

$$\text{Def}^\Box(\bar{\rho})(A) = \{(\rho, M, i, \beta)\} / \sim$$

where $(\rho, M, i)$ is as above, and $\beta$ is a basis for $M$ that lifts $\beta_F$ under $i$.

Equivalently, since we have fixed a basis, we see that

$$\text{Def}^\Box(\bar{\rho})(A) = \{\rho : G \to GL_n(A) : \rho \mod m_A = \bar{\rho}\}$$

$$\text{Def}(\bar{\rho})(A) = \text{Def}^\Box(\bar{\rho})(A) / (\text{conjugation by } \ker(GL_n(A) \to GL_n(k))).$$

Fact 1.3. One can check that

$$\text{Def}^\Box(\bar{\rho})(A) = \lim\limits_{\leftarrow} \text{Def}^\Box(\bar{\rho})(A/m_i^i A)$$

and similarly for $\text{Def}(\bar{\rho})$. What this tells us, is that we can simply study these functors on $\mathfrak{A}_\Lambda$. In particular, when we want to look at representability, we can just work in the category $\mathfrak{A}_{W(F)}$. Since if $R$ is a universal deformation ring for the category $\mathfrak{A}_\Lambda$, then for $A \in \mathfrak{A}_\Lambda$,

$$\text{Def}(\bar{\rho})(A) = \lim\limits_{\leftarrow} \text{hom}_{\Lambda} (R, A/m_i^i A)$$

and so $\text{Def}(\bar{\rho})$ is representable on $\hat{\mathfrak{A}}_\Lambda$. (Here, $A/m_i^i A$ are Artinian, because they are Noetherian of dimension 0, with $m_A/m_i^i A$ as the only prime ideal).
2. Mazur’s finiteness condition

**Definition 2.1.** A profinite group $G$ is said to satisfy the finiteness conditions $\Phi_p$ if, for all finite index open subgroup $H \subseteq G$, one of the following equivalent conditions hold:

1. The pro-$p$-completion of $H$ is topologically finitely generated
2. The abelianized pro-$p$-completion of $H$, given its natural $\mathbb{Z}_p$-structure, is of finite type over $\mathbb{Z}_p$
3. There are only a finite number of continuous homomorphisms from $H$ to $\mathbb{F}_p$ (finitely many open subgroups of index $p$)

**Example 2.2.**

1. $G_L$ where $L/\mathbb{Q}_p$ is a finite extension (think any local field is fine)

   \[ \text{Proof.} \text{ There are only finitely many extensions of a given degree (say degree } p) \]

2. $G_{K,S}$ where $K$ is any number field, $S$ is finite set of primes and $G_{K,S}$ is the Galois group of the maximal field extension which is unramified outside $S$

   \[ \text{Proof.} \text{ This is by Hermite-Minkowski theorem, which says that there are only finitely many extensions of } K \text{ of bounded degree unramified outside } S \]

3. Representability

   **Definition 3.1.** We say a functor $F : C \to \text{Sets}$ is representable, if there exists an object $R$ of $C$, such that

   \[ \Phi : \text{hom}(R, \cdot) \to F \]

   is a natural isomorphism.

   The $R$ is called an universal object. By Yoneda’s lemma, natural transformations from $\text{hom}(R, \cdot) \to F$ are in one to one correspondence with elements of $F(R)$. In particular, $\Phi_R(id_R)$ is called the universal element.

   **Theorem 3.2.** (Schlessinger’s Criterion). Suppose $F : \mathfrak{A}_{W(\mathbb{F})} \to \text{Sets}$ is a covariant functor with the property that $F(\mathbb{F})$ is a point. Suppose that for $A, B, C$ are objects of $\mathfrak{A}_{W(\mathbb{F})}$ with $A \to B$ and $B \to C$. Canonically, we have a map

   \[ \phi : F(A \times_C B) \to F(A) \times F(B) \]

   Suppose $F$ satisfies:

   1. $\phi$ is a surjection whenever $B \to C$ is a small morphism (surjective and its kernel is principal and annihilated by $m_A$)
   2. $\phi$ is a bijection when $C = \mathbb{F}$, and $B = \mathbb{F}[\epsilon]/\epsilon^2$
   3. $F \left( \mathbb{F}[\epsilon]/\epsilon^2 \right)$ is finite dimensional
   4. $\phi$ is a bijection whenever $A = B$; and $A \to C$ and $B \to C$ are small

   Then, $F$ is representable.

   **Remark 3.3.** As remarked by Brian Conrad in [7], fiber products in fact do not exists in $\hat{\mathfrak{A}}_\Lambda$, because the fiber product need not be Noetherian. This is why we work in the smaller category instead.

   \[ A = \mathbb{F}[X,Y], B = \mathbb{F} \text{ and } C = \mathbb{F}[X]. \] Let $A \to C$ by $Y \to 0$ and $B \to C$ in the canonical way. Then $A \times_C B = k \oplus Y \cdot k[X,Y]$. The maximal ideal is $Y \cdot k[X,Y]$, and so the tangent space is identified with $k[X]$ which is an infinite dimensional $k$-vector space. This implies that $A \times_C B$ is not Noetherian.

   **Theorem 3.4.** (Mazur). Assume that $G$ satisfies $\Phi_p$. Then

   1. $\text{Def}^\mathbb{F}$ is pro-representable by some $R_p^\mathbb{F} \in \hat{\mathfrak{A}}_\Lambda$
   2. If $\text{End}_{\mathfrak{F}[G]V_\mathbb{F}} = \mathbb{F}$ (which we will now say $\bar{\rho}$ is Schur), then $\text{Def}(\bar{\rho})$ is pro-representable by some $R_{\bar{\rho}} \in \hat{\mathfrak{A}}_\Lambda$
These are called the universal framed deformation ring and the universal deformation ring respectively.

Remark 3.5.
- Here, pro-representable means that when we view Def\(^{\square}\) as a functor from \(A_\Lambda\), it is representable by something in \(\hat{A}_\Lambda\).
- The condition \(\text{End}_{\bar{\rho} G} V_{\bar{\rho}} = \mathbb{F}\) is satisfied when \(\bar{\rho}\) is absolutely irreducible (ie, \(\bar{\rho} : G \to GL_n(\mathbb{F})\) is irreducible).
- Even if \(\Phi_p\) doesn’t hold, the universal ring \(R_{\bar{\rho}}^{\square}\) still exists as an inverse limit of Artinian rings, but it may not be Noetherian.

3.1. Explicit presentation for \(R_{\bar{\rho}}^{\square}\). This is taken from Proposition 1.3.1 of [1].

First, suppose that \(G\) is finite, with presentation
\[
\langle g_1, \ldots, g_s : r_1 (g_1, \ldots, g_s) = 1, \ldots, r_t (g_1, \ldots, g_s) = 1 \rangle.
\]
Then, let
\[
R = \Lambda \left[ X_{ij}^k : 1 \leq k \leq s, 1 \leq i, j \leq n \right] / I
\]
where \(I\) is the ideal generated by the coefficients of the matrices
\[
r_\ell \left( X^1, \ldots, X^n \right) - \text{Id}, \text{ for all } 1 \leq \ell \leq t,
\]
where \(X^k\) are the matrices \((X^k)_{1\leq i,j \leq n}\).

Basically, we just formed what the most canonical target space could be for any given representation. Now, we need to make sure that it is complete, local and Noetherian. Let \(J\) be the kernel of \(R \to \mathbb{F}\) defined by \(X^k \mapsto \bar{\rho}(g_k)\), and let \(R_{\bar{\rho}}^{\square}\) be the \(J\)-adic completion of \(R\). Let \(\bar{\rho}_{\text{univ}}\) be the representation mapping \(g_k\) to the image of \(X^k\) in \(GL_n(R_{\bar{\rho}}^{\square})\).

This is universal, because given \(\rho : G \to GL_n(A)\), we can define \(\phi : R_{\bar{\rho}}^{\square} \to A\) by mapping the entries of \(X^k\) to the corresponding entries of \(\rho(g_k)\). Then it is clear that \(\rho = \phi \circ \bar{\rho}_{\text{univ}}\) (technically, \(\text{Def}^{\square}(\bar{\rho})(\phi)\)).

Now, for the profinite groups \(G\), we can write \(G = \lim \pi G/H_i\), where \(H_i \subseteq \ker(\bar{\rho})\) are open normal subgroups. For each \(G/H_i\), we get a universal pair \((R_i^{\square}, \rho_i^{\square})\) and we just let
\[
\left( R_{\bar{\rho}}^{\square}, \rho_{\text{univ}}^{\square} \right) = \lim \left( R_i^{\square}, \rho_i^{\square} \right)
\]
which will satisfy the universal property. Now, it is clear that \(R_{\bar{\rho}}^{\square}\) will be complete and local, but perhaps not Noetherian. We can prove this by showing that it has a finite dimensional tangent space later.

3.2. Explicit Presentation for \(R_{\bar{\rho}}\). Originally, Mazur [7] proved the existence via Schlessinger’s criterion, which is not super explicit. We will present two more explicit constructions here due to Faltings, but presented in Darmon, Diamond, Taylor [3] (they said the proof is suggested by Faltings), and de Smit, Lenstra [4]. There is yet another proof by Kisin which can be found in [1] (I should probably cite the original source...).

First, the construction in DDT. Suppose \(g_1, \ldots, g_r\) are topological generators of \(G\), and let \(A_1, \ldots, A_r\) be some choice of lifts of \(\bar{\rho}(g_1), \ldots, \bar{\rho}(g_r)\) in \(M_n(\Lambda)\). Define a map
\[
i : M_n(\Lambda) \to M_n(\Lambda)^r
\]
by
\[
x \mapsto (xA_1 - A_1 x, \ldots, xA_r - A_r x).
\]
This has torsion free cokernel. Therefore,
\[
M_n(\Lambda)^r = i(M_n(\Lambda)) \oplus V
\]
for some submodule \(V \subseteq M_n(\Lambda)^r\).

Now, suppose \(\rho : G \to GL_n(R)\) is a lift of \(\bar{\rho}\). Let \(v_p = (\rho(g_1) - A_1, \ldots, \rho(g_r) - A_r) \in M_n(R)^r\) (recall that \(A_i \in M_n(\Lambda) \subseteq M_n(R)\) since \(R\) is a \(\Lambda\)-algebra). It is important to note that
Lemma 3.6. We say \( \bar{v} \) for some sending \( T \rho \) \( W \).

Remark 3.7. In Darmon, Diamond and Taylor, we can actually do this same construction with lifts of some type \( \chi \). By this, we mean \( \chi \) is a continuous character \( \chi : G \to \Lambda^\times \) and we are looking at the set of lifts with \( \det \rho = \chi \).

Now, for the construction given by de Smit and Lenstra. Let \( (R^\square_{\hat{\rho}}, \rho^\square_{\text{univ}}) \) be the universal object for the functor \( \text{Def}^\square \). Let \( R_{\hat{\rho}} \) be the subring of \( R^\square_{\hat{\rho}} \) generated by all the traces of \( \rho^\square_{\text{univ}} \).

Lemma 3.8. Suppose \( W \) is a representation of \( G \) over some ring \( A \in \hat{\mathfrak{A}}_\Lambda \) and suppose \( A' \subseteq A \) is an inclusion of rings in \( \mathfrak{A}_\Lambda \), so that \( A' \) has the induced topology of \( A \). Suppose \( A' \) contains all the traces of \( W \) given by the action of \( G \). Suppose that \( W \otimes_A A/m_A \) is absolutely irreducible (ie. \( \hat{\rho} \) is absolutely irreducible). Then there is an \( A' \) representation \( W' \) such that \( W' \otimes_{A'} A \cong W \) as \( A[G] \)-modules.

This lemma allows us to define a representation on \( R_{\hat{\rho}} \) from \( (R^\square_{\hat{\rho}}, \rho^\square_{\text{univ}}) \). Let \( \rho_{\text{univ}} \) be the equivalence class of this representation. We will now show \( (R_{\hat{\rho}}, \rho_{\text{univ}}) \) is universal for the functor \( \text{Def} \). Suppose \( \rho : G \to GL(V_A) \) is a lift of \( \hat{\rho} \). Choose some basis of \( V_A \) that lifts that of \( \hat{\rho} \)'s. By universality, there exists \( \phi \in \text{Hom}_A(R^\square_{\hat{\rho}}, A) \) such that

\[ \rho = \text{Def}^\square(\hat{\rho})(\phi) \circ \rho^\square_{\text{univ}}. \]

By restriction, we of course have a function \( \phi \in \text{Hom}_A(R_{\hat{\rho}}, A) \), and \( \text{Def}(\hat{\rho})(\phi) \circ \rho^\square_{\text{univ}} \) is the equivalence class of \( \rho \). For uniqueness, we know that such a map \( \text{Def}(\hat{\rho})(\phi) \) is uniquely determined on the traces of \( \rho^\square_{\text{univ}} \). Since these are dense in \( R_{\hat{\rho}} \), this is uniquely defined.

4. Even more explicit examples

Since Eric is in the audience, I assume that the explicit constructions above are not good enough as “examples”, so here is some more. Originally, this was a subsection. However, after some research, there seems to be a lot of awesome stuff to say, so it has been upgraded.
Example 4.1. (From lecture 3 of [2]) Let $K/\mathbb{Q}_p$ be a finite extension with $\ell \neq p$ and let $G = G_K$. Let $\bar{\rho}$ be the trivial representation of dimension $n$. Then $\text{End}_G \bar{\rho}$ is bigger than $k$, so we only know that the framed deformations are representable.

Suppose $\rho : G \to \text{GL}_n(R)$ is a deformation of $\bar{\rho}$, then the image of $G$ is $\Gamma_n R = \text{Id} + M_n(m_R) \subseteq \text{GL}_n(R)$ (congruence subgroup?). We can see that there is a pro-$p$ group isomorphism of $\Gamma_n(R)$ with the additive group $M_n(A)$. This means that $\rho$ factors through the maximal pro-$p$ quotient of $G$.

In particular, $\rho |_{I_K}$ factors through the $p$-part of the tame quotient $I_K^{\text{tame}} = I_K/I_K^{\text{wild}}$ of $K$. By the structure of local fields, we know that the $p$-part of $I_K^{\text{tame}}$ is

$$I_K^{\text{tame},(p)} = \mathbb{Z}_p(1).$$

Here, the twist is given by, if $\sigma \in I_K^{\text{tame},(p)}$ then $\text{Frob}_K \sigma \text{Frob}_K^{-1} = \sigma^q$ where $q = \#(O_K/m_K)$. Fix a lift $f \in G$ of $\text{Frob}_K$ and $\tau$ a topological generator of $I_K^{\text{tame},(p)}$. From this analysis, we see that any lift $\rho$ is uniquely defined by the imaged of $f$ and $\tau$ subject to the condition that

$$\rho(f)\rho(\tau) = \rho(\tau)^q \rho(f).$$

Therefore,

$$R^{\square}_\rho = \mathbb{A}[[\{f_{ij}, \tau_{ij}\}|_{1 \leq i, j \leq n}]]/I$$

where $I$ is the ideal generated by the coefficients of the relation

$$(\text{Id} + (f_{ij}))(\text{Id} + (\tau_{ij})) = (\text{Id} + (\tau_{ij}))^q (\text{Id} + (f_{ij})).$$

(Think you also have to complete the ring with respect to the kernel of $R^{\square}_\rho \to \mathbb{F}$).

Example 4.2. (From lecture 3 of [2]) Let $\tilde{\rho} : G \to \mathbb{F}^\times$ be a continuous character. Then $\tilde{\rho}$ has a Teichmuller lift $\hat{\rho} : G \to \mathbb{Z}_p^\times$. Then any lift of $\bar{\rho}$ is just $\hat{\rho}\chi$ where $\chi$ is the lift of the trivial character. So, without loss of generality, can assume that $\hat{\rho}$ is trivial.

Now, suppose $G = G_{Q,S}^{ab} \cong \prod_{\ell \in S} \mathbb{Z}_\ell$ with $p \in S$. Similar to the previous example, any lift $\rho$ factors through the maximal pro-$p$ quotient, which is

$$G_{Q,S}^{ab,(p)} = \prod_{\ell \in S, \ell \equiv 1 \mod p} (\mathbb{F}_\ell^\times)^{(p)} \times (1 + p\mathbb{Z}_p).$$

Then, $R^{\square}_\rho = \mathbb{A}[[G_{Q,S}^{ab,(p)}]]$ to be the formal group algebra over $\mathbb{A}$. Even more explicitly, this is

$$R^{\square}_\rho = \frac{\mathbb{A}[[\{X_{\ell} \in S, \ell \equiv 1 \mod p\}, T]]}{\left(\left\{(X_\ell + 1)^{p^{n_p(\ell - 1)} - 1}\right\}_{\ell \in S, \ell \equiv 1 \mod p}\right)}.$$

In particular, if $S = \{p, \infty\}$ then $R \cong \mathbb{A}[T]$.

Example 4.3. Let $\bar{\rho} : \hat{\mathbb{Z}} \to \text{GL}_2(\mathbb{F})$ by $1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Let

$$R = \mathbb{A}[X_{1,1}, X_{1,2}, X_{2,1}, X_{2,2}]$$

and let

$$\rho : \hat{\mathbb{Z}} \to \text{GL}_2(R)$$

be given by $1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{pmatrix}$.

Then this is clearly universal. Usually, $R^{\square}_\rho$ is of the form $R/I$ for some ideal $I$ that tells you the conditions that the generators have to satisfy. In our case, there are no conditions to impose.
5. Tangent Space

Recall that the relative tangent space of the local ring $R_{\bar{\rho}}$ is $\text{hom}_A (R_{\bar{\rho}}, \mathbb{F}[\epsilon])$ where $\mathbb{F}[\epsilon] = \mathbb{F}[X]/(X^2)$ is the ring of dual numbers. It is not hard to show that this is isomorphic to

$$\text{hom}_\mathbb{F} \left( m_{R_{\bar{\rho}}}/ \left( m_{R_{\bar{\rho}}}, m_{\Lambda} R_{\bar{\rho}} \right), \mathbb{F} \right).$$

By representability, we also see that this is just $\text{Def}(\bar{\rho}) (\mathbb{F}[\epsilon])$.

However, we have yet another interpretation. Given $V_{\mathbb{F}[\epsilon]} \in \text{Def}(\bar{\rho}) (\mathbb{F}[\epsilon])$, since $V_{\mathbb{F}[\epsilon]}/\epsilon V_{\mathbb{F}[\epsilon]} \cong V_{\epsilon}$, we get an exact sequence

$$0 \to \epsilon V_{\mathbb{F}[\epsilon]} \to V_{\mathbb{F}[\epsilon]} \to V_{\epsilon} \to 0.$$  

It is easy to check that $\epsilon V_{\mathbb{F}[\epsilon]} \cong V_{\epsilon}$, and so we see that $V_{\mathbb{F}[\epsilon]} \in \text{Ext}^1_{\Lambda[G]} (V_{\epsilon}, V_{\epsilon}) \cong H^1 (G, \text{Ad} \bar{\rho})$.

More explicitly, suppose $\rho \in \text{Def}(\bar{\rho}) (\mathbb{F}[\epsilon])$, we can write

$$\rho(g) = (1 + \epsilon c(g)) \bar{\rho}(g)$$

for some $c \in \text{Ad} \rho$, for all $g \in G$. Additionally,

$$\rho(gh) = (1 + \epsilon c(gh)) \bar{\rho}(gh)$$

$$\rho(g) \rho(h) = (1 + \epsilon c(g)) \bar{\rho}(g) (1 + \epsilon c(h)) \bar{\rho}(h).$$

This means that

$$c(gh) \bar{\rho}(gh) = c(g) \bar{\rho}(gh) + \bar{\rho}(g)c(h) \bar{\rho}(h)$$

$$c(gh) = c(g) + \bar{\rho}(g)c(h) \bar{\rho}(g)^{-1}$$

which means that $c$ is a 1-cocycle. Now, $\rho$ is isomorphic to $\rho'$ via conjugation by some $A = 1 + \epsilon B$ for some $B \in GL_n (\mathbb{F})$.

Then

$$\rho(g) = A \rho'(g) A^{-1} \text{ implies that}$$

$$\rho(g) = (1 + \epsilon B) (1 + \epsilon d(g)) \bar{\rho}(g) (1 - \epsilon B)$$

$$= \bar{\rho}(g) + \epsilon ((B + d(g)) \bar{\rho}(g) - \bar{\rho}(g)B)$$

This means that

$$c(g) = d(g) + B - \bar{\rho}(g)B \bar{\rho}(g)^{-1}.$$  

Therefore, $c - d$ is a 1-coboundary.

This whole thing also tells us that $\text{Def}_\mathbb{F} (\bar{\rho}) (\mathbb{F}[\epsilon]) \cong Z^1 (G, \text{Ad} \bar{\rho})$ is the set of 1-cocycles.

**Corollary 5.1.** Then $\dim_\mathbb{F} \text{Def}_\mathbb{F} (\bar{\rho}) (\mathbb{F}[\epsilon]) = \dim_\mathbb{F} \text{Def}(\bar{\rho}) (\mathbb{F}[\epsilon]) + n^2 - \dim_\mathbb{F} H^0 (G, \text{Ad} \bar{\rho}).$

**Proof.** This follows from the exact sequence

$$0 \longrightarrow (\text{Ad} \bar{\rho})^G \longrightarrow \text{Ad} \bar{\rho} \quad \overset{a \mapsto (g \mapsto g \cdot a \cdot a)}{\longrightarrow} \quad Z^1 (G, \text{Ad} \bar{\rho}) \longrightarrow H^1 (G, \text{Ad} \bar{\rho}) \longrightarrow 0.$$

**Proposition 5.2.** If $G$ satisfies $\Phi_p$, then $\text{Def}(\bar{\rho}) (\mathbb{F}[\epsilon])$ and $\text{Def}_\mathbb{F} (\bar{\rho}) (\mathbb{F}[\epsilon])$ are finite dimensional $\mathbb{F}$-vector spaces.

**Remark 5.3.** Technically, this immediately follows from the Noetherian-ness of $R_{\bar{\rho}}$, but we can show this more directly and illustrate why $\Phi_p$ is important.

**Proof.** Let $\bar{G} = \ker(\bar{\rho}) \subseteq G$ which is an open subgroup of $G$. By inflation restriction, we get

$$0 \to H^1 (G/\bar{G}, \text{Ad} \bar{\rho}) \to H^1 (G, \text{Ad} \bar{\rho}) \to (\text{hom}_{\mathbb{F}} \left( \bar{G}, \mathbb{F} \right) \otimes_{\mathbb{F}} \text{Ad} \bar{\rho})^{G/\bar{G}}$$

□
The left term is finite because $G/G$ and $\text{Ad} \bar{\rho}$ are finite. The term on the right is finite by the $\Phi_p$ assumption.

\section{Structure Theory of $R_p^\square$}

The forgetful functor induces a map $R_{\bar{\rho}} \to R_p^\square$, which turns out to be formally smooth. When $\text{End}_G(\bar{\rho}) = \mathbb{F}$, that means that $\dim \mathbb{F} H^0(G, \text{Ad} \bar{\rho}) = 1$. By Corollary 5.1, this implies that $R_p^\square$ is a power series ring over $R_{\bar{\rho}}$ of dimension $n^2 - 1$.

What’s going on here is that a universal deformation is just conjugation of the universal framed deformation by some residually trivial matrix. This conjugation is unique up to a unit scaling factor. To kill this unit scaling thing, we can demand that the upper left matrix entry is equal to 1. (See lecture 3 of [2])

Even more explicitly, let $\rho_{\text{univ}} : G \to GL_n(R_{\bar{\rho}})$ be some choice of the universal deformation (unframed). Let

$$\hat{\rho} : G \to GL_n\left(R_{\bar{\rho}}[[x_{ij}], i,j = 1,\ldots,n/(X_{1,1})]\right),$$

where $\hat{\rho} = (I d + X) \rho_{\text{univ}}(I d + X)^{-1}$ and $X$ is the matrix $X = (X_{ij})_{ij}$. The goal is to show that $\hat{\rho} = \rho_{\text{univ}}$.

Well, given $\rho : G \to GL_n(A)$, we have a map $\phi : R_{\bar{\rho}} \to A$ so that $\phi \circ \rho_{\text{univ}} \sim \rho$. Now, I just map $X_{ij}$’s to elements in $A$ that conjugates $\phi \circ \rho_{\text{univ}}$ to $\rho$. Uniqueness of this map is by the uniqueness of such a matrix (since we killed off the possible scaling).

\begin{theorem}
Suppose $r = \dim \mathbb{F} Z^1(G, \text{Ad} \bar{\rho})$ and $s = \dim \mathbb{F} H^2(G, \text{Ad} \bar{\rho})$. Then there is an $\Lambda$-algebra isomorphism

$$R_p^\square \cong \mathcal{O}[x_1,\ldots,x_r]/(f_1,\ldots,f_s).$$

\end{theorem}

\begin{proof}
See lecture 6 of [2].
\end{proof}

\section{Deformation Conditions}

In our definition of these deformation functors, we can make a restriction on the type of representations we can lift to. These are called deformation conditions. More specifically, we say $D$ is a deformation condition if the collection of lifts $(R, \rho)$ of $(\mathbb{F}, \bar{\rho})$ satisfies the following:

- $(\mathbb{F}, \bar{\rho}) \in D$
- Suppose $f : R \to S$ is a morphism in $\mathfrak{A}_\Lambda$, then $(S, f \circ \rho) \in D$
- Suppose $f : R \to S$ is a morphism in $\mathfrak{A}_\Lambda$, then $(R, \rho) \in D$ iff $(S, f \circ \rho) \in D$
- Suppose $R_1, R_2 \in \text{ob}(\mathcal{C}_\mathcal{O}), I_1, I_2$ ideals of $R_1, R_2$ respectively, such that there is an isomorphism $f : R_1/I_1 \cong R_2/I_2$. Suppose $(R_1, \rho_1), (R_2, \rho_2) \in D$ with $f(\rho_1 \mod I_1) = \rho_2 \mod I_2$ then

$$\{(a, b) \in R_1 \oplus R_2 : f(a \mod I_1) = b \mod I_2, \rho_1 + \rho_2\} \in D.$$

- If $I_1 \supseteq I_2 \supseteq \ldots$ is a sequence of ideals of $R$ with $\bigcap_j I_j = (0)$ and $(R/I_j, \rho \mod I_j) \in D$ for all $j$, then $(R, \rho) \in D$
- If $(R, \rho) \in D$ and $a \in \ker(GL_n(R) \to GL_n(\mathbb{F}))$ then $(R, \rho a^{-1}) \in D$

\begin{remark}
The point of these conditions is to be able to view deformation with additional conditions, as a subfunctor that is still representable. To get this second part, we need the set of lifts to still satisfy Schlessinger’s criterion, which is why we have these properties.

The geometric way to look at it, is that we have a surjective map $R_{\bar{\rho}} \to \mathbb{T}$ of the universal deformation ring to some appropriate Hecke algebra. That is, $\text{Spec} \mathbb{T}$ is a closed subspace of the space of deformations, which includes the closed point $\bar{\rho}$. The goal of modularity lifting is to impose more conditions to cut out the “modular” locus. From this point of view, we want the these more restrictive representations to be a closed subscheme of $\text{Spec} R_{\bar{\rho}}$. More specifically, we would like there to exists an ideal $I$ of $R_{\bar{\rho}}$ such that for all $f : R \to A$, $f \circ \rho_{\text{univ}}$ satisfies the deformation condition iff $f$ factors through $R/I$. We will see this is indeed the case with the next theorem.

Note that for any $a \in \ker(GL_n(\mathbb{R}_p^\square) \to GL_n(\mathbb{F}))$, it acts on $\mathbb{R}_p^\square$ via the universal property $\rho^\square \mapsto a^{-1} \rho^\square a$ (I am not sure why it’s written this way instead of $a \rho^\square a^{-1} \ldots$). Note: this is not a group action.
Proposition 7.2.

1. If \( D \) is a deformation problem, then there is a kernel \( (GL_n(R^\square_p) \to GL_n(F)) \)-invariant ideal \( I(D) \subseteq R^\square_p \) such that \((R, \rho) \in D\) if and only if the map \( R^\square_p \to R \) induced by \( \rho \) factors through the quotient \( R^\square_p/I(D) \).

2. Let \( \hat{L}(D) \subseteq \mathbb{Z}^1(G, \text{ad}\hat{\rho}) \cong \hom\left(\mathfrak{m}_{R^\square_p}/(\lambda, \mathfrak{m}_{R^\square_p}^2), F\right) \) be the annihilator of the image of \( I(D) \) in \( \mathfrak{m}_{R^\square_p}/(\lambda, \mathfrak{m}_{R^\square_p}^2) \). Then \( \hat{L}(D) \) is the preimage of some subspace in \( H^1(G, \text{ad}\hat{\rho}) \). (to be honest, I fail to see why this is an important result)

3. If \( I \) is a kernel \( (GL_n(R^\square_p) \to GL_n(F)) \)-invariant radical ideal of \( R^\square_p \), and \( I \neq \mathfrak{m}_{R^\square_p} \), then
\[
D(I) = \left\{ (R, \rho) : \frac{R^\square_p}{\mathfrak{m}_{R^\square_p}} \to R \text{ factors through } \frac{R^\square_p}{I} \right\}
\]
is a deformation problem. Furthermore, we have \( I(D(I)) = I \) and \( D(I(D)) = D \).

Now, for some examples of deformation problems.

7.1. Fixing determinants. Let \( \chi : G \to \Lambda^\times \) be a character, with \( \chi \mod p = \det \hat{\rho} \). We can consider the set of lifts with determinant \( \chi \).

Let \( \chi^{uni} : G \to (R^\square_p)^\times \) be the determinant of the universal deformation. Let \( I \) be the ideal generated by \( \chi^{uni}(g) - \chi(g) \).

Then \( \frac{R^\square_p}{I} \) represents the deformations with determinant \( \chi \). The similar results hold for unframed deformations.

Proposition 7.3. The tangent space of the set of deformations of fix determinant is isomorphic to \( H^1(G, \text{ad}^0 \hat{\rho}) \) (trace zero endomorphisms).

7.2. Unramified condition. Let \( K \) be a global field, and let \( S \) be a finite set of primes of \( K \). Let \( G = G_{K,S} \) be the Galois group of the maximal Galois extension of \( K \) unramified outside \( S \). Let \( \rho \) be a residual representation of \( G \) which is also unramified at some \( \nu \in S \).

We can consider the set of lifts \( \rho \) that are unramified at \( \nu \) (that is, \( \rho |_{G_{K,\nu}} \) is unramified for any choice of decomposition group \( G_{K,\nu} \)). This is of course the iff \( \rho \) is trivial on the inertia group \( I_{K,\nu} \). Let \( J \) be the ideal of \( R^\square_p \) generated by the entries of \( Id - \rho^{uni}(g) \) for all \( g \in I_{K,\nu} \). Then \( \frac{R^\square_p}{J} \) represents the unramified condition.

7.3. Ordinary deformations. There are many definitions of this. Let \( K/\mathbb{Q}_p \) be a finite extension and let \( G = G_K \). Let \( \psi : G \to \mathbb{Z}_p^\times \) be the \( p \)-adic cyclotomic character. Then \( \rho : G \to GL_n(R) \) is ordinary if
\[
\rho |_{I_K} \sim \begin{pmatrix}
\psi e_1 & * & * & * \\
0 & \psi e_2 & * & * \\
0 & 0 & ... & * \\
0 & 0 & 0 & \psi e_{n-1}
\end{pmatrix}
\]
where \( e_1 > e_2 > ... > e_{n-1} > 0 \).

REFERENCES