AUTOMORPHIC L-FUNCTIONS NOTES

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Abstract. These are notes I took in class, taught by Professor Ngo Bao Chau. I claim no credit to the originality of the contents of these notes. Nor do I claim that they are without errors, nor readable.

Full course notes available at math.uchicago/~ngo/GJ.pdf

1. Introduction and Overview

Definition 1.1. Define the Riemann zeta function to be
\[ \zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}. \]
Both sides of these converge absolutely on \( \text{Re}(s) > 1 \).

Riemann found that \( \zeta(s) \) can be extended to a meromorphic function of \( s \in \mathbb{C} \) with a single simple pole at \( s = 1 \). It satisfies a functional equation of the following form. Let \( \xi(s) = \pi^{s/2} \Gamma(\frac{s}{2}) \zeta(s) \) then
\[ \xi(s) = \xi(1 - s). \]

Tate’s Thesis uses Fourier analysis on adeles to explain more about this Euler product. For local fields \( \mathbb{R} \) and \( \mathbb{Q}_p \), they are locally compact groups. Therefore, they have invariant measures on them, which gives us Fourier analysis.

Definition 1.2. Define \( \mathbb{A}_\mathbb{Q} = \mathbb{R} \times \prod'_p \mathbb{Q}_p \), where the restricted product means that \( x = (x_v) \in \mathbb{A}_\mathbb{Q} \) is such that \( x_v \in \mathcal{O}_v \) for almost all \( v \). We know that \( \mathbb{A} \) is locally compact, so also have a Haar measure.

Definition 1.3. For a locally compact abelian group \( A \), define the Pontryagin dual to be
\[ \hat{A} = \{ x : A \to \mathbb{C}^1 \text{ unitary characters : continuous homomorphism} \}. \]
We will call \( \chi : A \to \mathbb{C}^\times \) a character.

The dual \( \hat{A} \) will be equipped with the compact open topology, making it a locally compact abelian group.

Theorem 1.4. (Pontryagin Theorem) \( A \cong \hat{\hat{A}} \).

Example 1.5. For \( A = \mathbb{Z} \), we know that \( \hat{A} = \mathbb{C}^1 \). Then the theorem says that all \( \chi : \mathbb{C}^1 \to \mathbb{C}^1 \) is of the form \( \chi(x) = x^n \) for some \( n \in \mathbb{Z} \).

Example 1.6. Suppose \( A = F \) is a local field. Then \( \hat{A} = F \) (not canonical). This is because for \( x \in F \), we can define \( \chi_x : F \to \mathbb{C}^1 \) given by \( y \to \psi_1(xy) \) where \( \psi_1 : F \to \mathbb{C}^1 \) is some preferred character. When \( F = \mathbb{R} \), let \( \psi_1(x) = \exp(2\pi ix) \).

Example 1.7. When \( A = \mathbb{A}_\mathbb{Q} \), then \( \hat{A} = \mathbb{A} \) as well.

Definition 1.8. For \( \varphi \in L^1(A) \), define \( \hat{\varphi} : \hat{A} \to \mathbb{C} \) be
\[ \hat{\varphi}(\chi) = \int_A \varphi(x) \chi(x)^{-1} \, dx \]
for some chosen Haar measure \( dx \). Then \( \hat{\varphi} \) is continuous and “0 at infinity”.

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**Definition 1.9.** When $A = F$, define the Schwartz space to be

$$\mathcal{S}(A) = \begin{cases} F = \mathbb{R} & \text{smooth, rapid decay, like } p(x)e^{-x^2} \\ F = \mathbb{Q}_p & \text{locally unipotent compact support, like } 1_{\mathbb{Z}_p}. \end{cases}$$

The functions in $\mathcal{S}(F)$ are stable under Fourier transform.

Note, when we have a Haar measure on $A$, we have it on $\hat{A}$ as well. By using Fourier inversion,

$$\int_{\hat{A}} \hat{\varphi}(\chi) d\chi = \varphi(0).$$

For $A = F$, there exists a unique $dx$ self-dual so that

$$\int_{F} \hat{\varphi}(x) dx = \varphi(0).$$

For $b \in \mathcal{S}(F)$, $\hat{b} = b$ self-dual. When $F = \mathbb{R}$, let $b_{\infty}(x) = e^{-\pi x^2}$. When $F = \mathbb{Q}_p$, let $b_p = 1_{\mathbb{Z}_p}$. These are called basic functions.

We have $\mathcal{S}(A_{\mathbb{Q}}) = \{\text{finite linear combination of functions } \varphi = \bigotimes_v \varphi \text{ where } \varphi_v \in \mathcal{S}(F_v) \text{ and } \varphi_v = b_v \text{ for almost all } v\}$.

**Theorem 1.10.** (Poisson Summation formula). For all $\varphi \in \mathcal{S}(A)$,

$$\sum_{\alpha \in k} \varphi(\alpha) = \sum_{\alpha \in k} \hat{\varphi}(\alpha).$$

Tate applies Mellin transform to this whole package. Define

$$\Omega(F^\times) = \{\chi : F^\times \to \mathbb{C}^\times : \text{continuous, homomorphic}\}.$$

We have module character $F^\times \to \mathbb{R}_+$ (absolute value). For $\alpha \in F^\times$, $d(\alpha x) = |\alpha| dx$. We have a homomorphism $\mathbb{C} \to \Omega(F^\times)$ given by

$$s \mapsto (x \mapsto |x|^s).$$

The image $\Omega^0(F^\times)$ is neutral connected component of $\Omega(F^\times)$. The map

$$\mathbb{C} \to \Omega^0(F^\times)$$

is an isomorphism when $F = \mathbb{R}$, and kernel $2\pi i \log p$ if $F = \mathbb{Q}_p$ (second one may be a bit off).

**Definition 1.11.** Mellin transform. Start with $\varphi \in \mathcal{S}(F) \mapsto M_\varphi : \Omega(F^\times) \to \mathbb{C}$ where

$$M_\varphi(\chi) = \int_{F^\times} \varphi(x) \chi(x)^{-1} d^\times x$$

where $d^\times x = \frac{dx}{|x|}$ is a Haar measure on $F^\times$.

Notice that this map factors through the restriction to $F^\times$. However, note that $\varphi \mid_{F^\times}$ no longer has rapid decay, but $M_\varphi$ is a meromorphic function on $\Omega(F^\times)$. When $F^\times = \mathbb{Q}_p^\times$, $M_\varphi$ has a simple pole at $s = 0$.

**Example 1.12.** For $F = \mathbb{Q}_p$,

$$M_{b_\infty}(\chi) = \begin{cases} 0 & \text{if } \chi \notin \Omega^0(F^\times) \\ (1 - p^{-s})^{-1} & \text{if } \chi = |\cdot|^s. \end{cases}$$

For $F = \mathbb{R}$,

$$M_{b_{\infty}}(\chi) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right).$$

Question: what is the effect of Fourier transform on the Mellin transform? How to relate $M_{\hat{\varphi}}$ with $M_\varphi$? 
Consider $\varphi \in \mathcal{S}(F)$, maps to $\varphi_a(x) = \varphi(ax)$ for some $a \in F^\times$. Then $\hat{\varphi}_a = |a|^{-1} \hat{\varphi}_{1/a}$. This is the analog of $s \leftrightarrow 1 - s$ (the 1 is the $|a|^{-1}$).

We can compute that

$$M \hat{\varphi}(\chi) = \gamma(\chi) M \varphi \left( |\chi|^{-1} \right),$$

where $\gamma$ is a factor depending solely on $\chi$ and not on $\varphi$. This is called the Tate local functional equation.

$$\hat{\varphi}(y) = \int \hat{\varphi}(x^{-1}) e(xy) dx$$

where $\hat{\varphi}(x) = \varphi(x^{-1})$. We write it this way, because it looks like a convolution. This will behave well with the Mellin transform.

Mellin transform for $A$. Suppose $\varphi \in \mathcal{S}(A)$, we want to map it to $M \varphi : \Omega(K^\times \backslash A^\times) \to \mathbb{C}$. First, we have $|.| : A^\times \to \mathbb{R}_+$. Let $A^1$ be the kernel of $|.|$, then $K^\times \backslash A^1$ is compact. Similar to the above, we have

$$\mathbb{C} \to \Omega \left( K^\times \backslash A^\times \right)$$

by

$$s \mapsto (x \mapsto |x|^s)$$

and $\mathbb{C} \to \Omega^0 \left( K^\times \backslash A^\times \right)$.

First, suppose $\varphi = b_\infty \otimes b_p$. Then

$$M \varphi(s) = \pi^{-s/2} \Gamma \left( \frac{s}{2} \right) \prod_p (1 - p^{-s})^{-1}$$

for $\text{Re}(s) > 1$. For every $\varphi \in \mathcal{S}(A)$, define

$$M \varphi : \Omega(K^\times \backslash A^\times) \to \mathbb{C}$$

holomorphic for $\text{Re}(\chi) > 1$. Can show that $M \varphi$ can be meromorphically continued to all of $\Omega(K^\times \backslash A^\times)$ satisfying

$$M \hat{\varphi}(\chi) = M \varphi \left( |\chi|^{-1} \right).$$

This is proved with Poisson summation formula, which comes together with the meromorphic continuation. The functional equation is the same as saying the product of the $\gamma$-factors

$$\prod \gamma_v(\chi) = 1.$$ 

However, this never converges, so does not really make sense.

Note, $\Omega(K^\times \backslash A^\times)$ is automorphic forms of $GL_1$. We will also talk about generalizations of this due to Langlands, to automorphic forms on any reduction group $G/K$. These are irreducible representation of $G(A)$, which is

$$\pi \subseteq \mathbb{A}(G(Q) \backslash G(A))$$

which breaks down into $\pi = \bigoplus_v \pi_v$ where $\pi_v$ is unramified for almost all $v$ and $\pi'_v(G(O_v)) = 0$.

Langlands define a dual group $\hat{G}/\mathbb{C}$ whose root system is dual to that of $G$, for $G$ split. These unramified representations are the same as semisimple conjugacy class of $\alpha_p \in \hat{G}$. With these datum, we can define $L$-functions.

Let $\varphi$ be a representation

$$\varphi : \hat{G} \to GL(n).$$

Define

$$L(s, \pi, \rho) = \prod_v L(s, \pi_v, \rho)$$

where

$$L(s, \pi_v, \rho) = \begin{cases} \det (1 - \rho(\alpha_p) p^{-s})^{-1} & \text{\pi}_v \text{ unramified} \\
\text{hard...} & \text{else} \end{cases}$$
Langlands showed that $L(s, \pi, b)$ converges for $\Re(s) > N$. It is conjectured that it can be meromorphically continued and has a functional equation.

This functional equation is very important. When we are given a general representation, its associated $L$-function has these functional equation properties if the representation is automorphic.

These are known for $G = GL_n$ and $\rho$ is standard representation (by Godement-Jacquet). There is another conjecture of Langlands that all other cases can be reduced to this one (functoriality).

2. Adèles and Idèles

2.1. For $\mathbb{Q}$.

Theorem 2.1. (Ostrowski) There are no other norms on $\mathbb{Q}$ other than $|\cdot|_\infty$ and $|\cdot|_p$ (up to equivalence).

Proposition 2.2. (Product formula) For $x \in \mathbb{Q}$, $|x|_\infty \prod_p |x|_p = 1$.

Proof. For $x \in \mathbb{N}$, we can write $x = p_1^{r_1} \ldots p_n^{r_n}$. Then $|x|_p = p_i^{-r_i}$ and $|x|_\infty = x$. \qed

Definition 2.3. Adèles over $\mathbb{Q}$ is defined to be

$$A_{\mathbb{Q}} = \prod_v \mathbb{Q}_v = \{(x_\infty, x_p, \ldots) : x_\infty \in \mathbb{R}, x_p \in \mathbb{Q}_p, x_p \in \mathbb{Z}_p \text{ for a.e. } p\}.$$ 

Notice that $\mathbb{R}$ and $\mathbb{Q}_p$ are locally compact, with $[-1, 1]$ and $\mathbb{Z}_p$ compact neighbourhood of 0. Notice that

$$[0, 1] = \left[0, \frac{1}{n}\right] \cup \ldots \cup \left\{\frac{n-1}{n}, 1\right\}, \text{ and } \mathbb{Z}_p = \bigcup_{r \in \mathbb{Z}} (r + p\mathbb{Z}_p).$$

Similar, on $A_{\mathbb{Q}}$, we have

$$A_{\mathbb{Q}} = \bigcup_{N \in \mathbb{N} \mathbb{R} \times \{(x_p)_p : N x_p \in \mathbb{Z}_p\} = \bigcup\mathbb{N} \mathbb{R} \times N^{-1}\mathbb{Z}}.$$ 

We give $A_{\mathbb{Q}}$ the finest topology such that $\mathbb{R} \times N^{-1}\mathbb{Z} \hookrightarrow A$ is continuous, where $\mathbb{R} \times N^{-1}\mathbb{Z}$ is given the product topology. Then

$$[-1, 1] \times \mathbb{Z}$$

is a compact neighbourhood of 0 (by Tychonov’s theorem, $\mathbb{Z}$ is compact). There is a canonical embedding $\mathbb{Q} \hookrightarrow A$ diagonally.

Theorem 2.4. $\mathbb{Q}$ is a discrete co-compact subgroup of $A$. Moreover, there is an exact sequence of topological group

$$0 \rightarrow \mathbb{Z} \rightarrow A/\mathbb{Q} \rightarrow \mathbb{R}/\mathbb{Z} \rightarrow 0.$$ 

In other words, $A/\mathbb{Q}$ is the pro-universal covering of the circle.

$$\lim\mathbb{R}/\mathbb{Z}$$

where $\mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ is given by $x \mapsto nx$.

Proof. Let $B = (-1, 1) \times \mathbb{Z}$. Then

$$B \cap \mathbb{Q} = \{x \in \mathbb{Q} : x \in (-1, 1), x \in \mathbb{Z}_p \text{ for all } p\} = (-1, 1) \cap \mathbb{Z} = \{0\}.$$ 

This gives the commutative diagram

$$\begin{array}{ccccccc}
0 & \rightarrow & \mathbb{R} \times \mathbb{Z} & \rightarrow & A & \rightarrow & \bigoplus_p \mathbb{Q}_p/\mathbb{Z}_p & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & 0 & \rightarrow & \mathbb{Q} & \rightarrow & \mathbb{Q} & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} \\
\end{array}$$
By the Snake lemma,

\[ 0 \to \mathbb{Z} \xrightarrow{\text{diagonal}} \hat{\mathbb{Z}} \times \mathbb{R} \to \mathbb{A}/\mathbb{Q} \to 0 \]

and so \( \mathbb{A}/\mathbb{Q} \cong \left( \hat{\mathbb{Z}} \times \mathbb{R} \right)/\mathbb{Z} \) and so

\[ 0 \to \hat{\mathbb{Z}} \to \mathbb{A}/\mathbb{Q} \to \mathbb{R}/\mathbb{Z} \to 0. \]

□

There is another proof for compactness of \( \mathbb{A}/\mathbb{Q} \). It is enough to construct a compact subset \( B \subseteq \mathbb{A} \) mapping surjectively in the map \( \mathbb{A} \to \mathbb{A}/\mathbb{Q} \). Take

\[ B = \{ (x_\infty, x_p) : |x_\infty| \leq 1, |x_p|_p \leq 1 \}, \]

which is compact. By the same exact sequence,

\[ B + \mathbb{Q} \to \mathbb{A} \]

is surjective, and so \( \mathbb{A}/\mathbb{Q} \) is compact. It is important to note that here, \( B \) is just a set, not a group.

**Definition 2.5.** Define the idèles \( \mathbb{A}^\times \) to be

\[ \mathbb{A}^\times = \{ (x_\infty, x_p) : x_\infty \in \mathbb{R}^\times, x_p \in \mathbb{Q}_p^\times, x_p \in \mathbb{Z}_p^\times \text{ for almost all } p \}. \]

This is also a locally compact abelian group, by giving the topology where \( \mathbb{A}^\times \to \mathbb{A} \) given by \( x \mapsto x \) and \( x \mapsto x^{-1} \) are both continuous.

Consider the map \( \mathbb{Q}^\times \hookrightarrow \mathbb{A}^\times \xrightarrow{|\cdot|} \mathbb{R}_+^\times \) where \( |x|_\mathbb{A} = \prod_v |x_v|_v \). Define \( \mathbb{A}^1 \) to be \( \ker |\cdot|_\mathbb{A} \subseteq \mathbb{A} \), which contains \( \mathbb{Q}^\times \) by the product formula.

**Theorem 2.6.** \( \mathbb{Q}^\times \) is a discrete cocompact subgroup of \( \mathbb{A}^1 \).

**Proof.** Similar to the additive case. The claim is the

\[ \mathbb{Q}^\times \times \hat{\mathbb{Z}}^\times \times \mathbb{R}_+^\times \to \mathbb{A}^\times \]

is an isomorphism (the map is given by multiplication since each is a subgroup of \( \mathbb{A}^\times \)). Then

\[ \mathbb{A}^1/\mathbb{Q}^\times \cong \hat{\mathbb{Z}}^\times \]

is compact. □

**Remark 2.7.** For general number fields, everything extends the same way, except for this theorem. It becomes quite hard and fundamental.

2.2. **General number fields.** Let \( K \) be a number field over \( \mathbb{Q} \).

**Theorem 2.8.** (Ostrowski). For all rational primes \( p \),

\[ K \otimes_\mathbb{Q} \mathbb{Q}_p = \prod_{v \mid p} K_v. \]

Similarly,

\[ k_\infty = k \otimes_\mathbb{Q} \mathbb{R} = \mathbb{R}^\times \otimes \mathbb{C}^\times. \]

Let \( x \in K \), define \( |x|_v = |N_{K_v/\mathbb{Q}_p}(x)|_v \). Then these are all the absolute values on \( K \).

**Proposition 2.9.** (Product formula) For all \( x \in K^\times \),

\[ \prod_v |x|_v = 1. \]
Definition 2.10. Define

$$A_K = \prod_v K_v = \{(x_v) : x_v \in K_v, x_v \in \mathcal{O}_v \text{ for almost all finite } v\}.$$ 

Similarly, define

$$A_K^\times = \{(x_v) : x_v \in K_v^\times, |x_v| = 1 \text{ for almost all finite } v\}.$$ 

Theorem 2.11. The embedding $K \hookrightarrow A_K$ diagonally is a discrete cocompact group. Since $A_K = A_Q \otimes Q K$, we have

$$A_K / K \cong (A_Q / Q)^r,$$

where $r = [K : Q]$.

Similarly, $K^\times$ is a discrete cocompact subgroup of $A_K^1$.

Remark 2.12. This will imply the finiteness of class number, and Dirichlet’s unit theorem.

Proof. Discreteness is similar to the additive case. Use the box

$$B_c = \{(x_v) : |x_v| \leq c_v\}$$

for some $c = (c_v)$ and $c_v \in \mathbb{R}^+_\times$, $c_v = 1$ for almost all $v$. For all $c$ as above, $K^\times \cap B_c$ is finite.

Lemma 2.13. (Minkowski) There exists a constant $C > 0$ depending only on the discriminant of $K/Q$, such that for all $c = (c_v)$, $c_v \in \mathbb{R}^+_\times$, $c_v = 1$ for almost all $v$, such that $\prod c_v > C$, then $K^\times \cap B_c \neq \emptyset$.

Lemma 2.14. For all sequences $c = (c_v)$ as above, $B_c \cap A_K^1$ is compact.

Proof. For $(x_v) \in B_c$, $|x_v| \leq c_v$. For $x \in B_c \cap A_K^1$, we have

$$|c|^{-1} c_v \leq |x_v| \leq c_v$$

where $|c| = \prod c_v$. Then

$$B_c \cap A_K^1 \subseteq \prod U_v$$

where $U_v \subseteq K_v^\times$ is

$$U_v = \left\{ x_v \in K_v^\times : |c|^{-1} c_v \leq |x_v| \leq c_v \right\}.$$ 

Since for almost all $c$, $c_v = 1$, we have

$$|c|^{-1} \leq |x_v| \leq 1.$$ 

When $q_v > c_v$ (the size of the residue field), the inequality implies that $|x_v| = 1$. That is, $U_v = \mathcal{O}_v^\times$. \hfill \Box

For $|c| > C$ from Minkowski,

$$A_K^1 \subseteq \cup_{\alpha \in K^\times} \alpha B_c$$

and $\alpha B_c \cap A_K^1$ compact. Hence, $A_K^1 / K^\times$ is compact. \hfill \Box

From the theorem, we get an exact sequence

$$0 \rightarrow \left( \mathcal{O}_K^\times \times K_1^\times \right) / \mathcal{O}_K^\times \rightarrow A_K^1 / K^\times \rightarrow Cl_K \rightarrow 0$$

where $K_1^\times$ are the norm one elements are the infinite places. Since $Cl_K$ is discrete, and $A_K^1 / K^\times$ is compact, this implies that $K_1^\times / \mathcal{O}_K^\times$ is compact. That is, $K_1^\times$ and $\mathcal{O}_K^\times$ have the same rank.

Let $F$ be a local field. Given $a, b \in \mathbb{R}^+_\times$, the sets

$$F_{a,b}^\times = \{ x \in F^\times : a \leq |x| \leq b \}$$

is compact. For $K$ a global field, $v$ places, $a_v, b_v \in \mathbb{R}^+_\times$ the sets

$$\prod_v F_{a,b}^\times$$
is compact by Tychonoff’s theorem. It is however unclear whether or not
\[ \prod_v F_{a,b}^\times \cap A^\times \]
is compact in $A^\times$. Unless $F_{a,b}^\times = \mathbb{Z}_p^\times$ for almost all $v$, there’s no way that this set is compact. This is because it can be written as infinite disjoint union of open subsets. This is because a system of open neighbourhoods of $A^\times$ is
\[ \prod_{v \in S} U_v \times \prod_{v \notin S} \mathbb{Z}_v^\times \]
for some finite set $S$, where $U_v \subseteq F_v^\times$ is some open subset. However, $\prod_v F_{a,b}^\times \cap A^\times$ is compact is $F_{a,b}^\times = \mathbb{Z}_p^\times$ for almost all $v$.

2.3. Haar Measure. Let $F_v$ be a local field. Pick $dx$ a Haar measure on $F_v$ self-dual with the choice of a preferred character $\psi_F : F \to \mathbb{C}$. We have a linear form $C_c(F_v) \to \mathbb{C}$ given by
\[ \varphi \mapsto \int_F \varphi(x) dx. \]

For adèles it also has a Haar measure $dx = \prod_v dx_v$. We also have a linear form $C_c(A) \to \int_A \varphi(x) dx$. We map
\[ \varphi = \otimes_v \varphi_v \in C_c(A) \mapsto \prod_v \int_{F_v} \varphi_v(x) dx \]
where $\varphi_v = 1_{\mathcal{O}_v}$ for almost all $v$.

Locally, for $dx$ a Haar measure on $F_v$, $\frac{d x}{|x|}$ is our choice of Haar measure on $F_v^\times$. Then
\[ \frac{d(xy)}{|xy|} = \frac{|y|}{|x|} \frac{dx}{|x|} = \frac{dx}{|x|} \]
so it is invariant under multiplication. However,
\[ \prod_v \frac{dx_v}{|x_v|} \]
is not a Haar measure on $A^\times$. Suppose $\varphi = \otimes_v \varphi_v$ with $\varphi_v = 1_{\mathcal{O}_v}^\times$ for almost all $v$. When $\varphi_v = 1_{\mathcal{O}_v}^\times$,
\[ \int_{F_v^\times} \varphi_v(x) \frac{dx}{|x|} = \int_{F_v^\times} \varphi_v(x) dx = \frac{p-1}{p} \]
so the infinite product goes to 0, which is bad.

2.3.1. Review on Pontryagin duality. Generalization of Fourier series and Fourier transform to all locally compact abelian group.

Let $G$ be a locally compact abelian group.

Example 2.15. Take $G = \mathbb{Z}, \mathbb{C}, F, F_v, ...$

Let $C_c(G)$ be the continuous functions with compact support. We say $f_n \to f$ if for every compact $K \subseteq G$, $f_n |_K \to f |_K$ uniformly. Then
\[ C_c(G) = \bigcup_{K \subseteq G} C_K(G). \]

A random measure is $G$ is a continuous linear form $\mu : C_c(G) \to \mathbb{C}$. Define $\ell_x(\varphi)(y) = \varphi(x^{-1} y)$.

Definition 2.16. The measure $\mu$ is said to be left invariant, if for all $x \in G$,
\[ \mu(\ell_x \varphi) = \mu(\varphi) \]
for all $\varphi \in C_c(G)$.

Theorem 2.17. (Haar, Von Neuman) Every locally compact abelian group has a left invariant measure, unique up to a non-zero scalar.
Definition 2.18. Let
\[
\Omega(G) = \{ \chi : G \to \mathbb{C}^\times \text{ continuous} \}
\]
be the set of cocharacters of \( G \) and
\[
\Lambda(G) = \{ \chi : G \to \mathbb{C}^1 \text{ continuous} \}
\]
be the set of unitary characters.

These groups are topological groups with topology the compact-open topology. \( \Lambda(G) \) as a topological abelian group is the Pontryagin dual of \( G \).

Example 2.19. \( \Lambda(\mathbb{Z}) = \mathbb{C}^1 \) and \( \Lambda(\mathbb{C}^1) = \mathbb{Z} \).
\( \Lambda(\mathbb{R}) = \mathbb{R} \) (identification by a preferred character), \( \Lambda(\mathbb{Q}_p) = \mathbb{Q}_p \) and \( \Lambda(\mathbb{A}) = \mathbb{A} \).

Meanwhile, \( \Lambda(\mathbb{R}^\times) = \Lambda(\pm \times \mathbb{R}^\times) = \{ \pm 1 \} \times \Lambda(\mathbb{R}^\times) = \{ \pm 1 \} \times \mathbb{R} \).

Definition 2.20. Fourier transform Let \( \varphi \in L^1(G, dx) \). Define \( \hat{\varphi} : \Lambda(G) \to \mathbb{C} \) by
\[
\hat{\varphi}(\chi) = \int_G \varphi(x) \chi(x)^{-1} dx.
\]

Theorem 2.21. (Riemann-Lebesgue) For \( \varphi \in L^1(G, dx) \) then \( \hat{\varphi} \) is a continuous function tending to 0 as \( \chi \to \infty \).

Remark 2.22. This means the following. For all \( a > 0 \), let
\[
K_a = \{ \chi \in \Lambda(G) : |\hat{\varphi}(\chi)| \geq a \}.
\]

We mean that for all \( a > 0 \), \( K_a \) is compact.

For any \( X \) locally compact topological space, we can do the one point compactification by \( X \cup \{ \infty \} \). A system of neighbourhood of \( \infty \) is declared to be the complement of compact subsets of \( X \).

We say \( \hat{\varphi} : \Lambda(G) \to \mathbb{C} \) tends to 0 as \( \chi \to \infty \) by \( \hat{\varphi} \) extended to \( \Lambda(G) \cup \{ \infty \} \) by \( \hat{\varphi}(\infty) = 0 \) is continuous. Here, we are assuming the following theorem.

Theorem 2.23. If \( G \) is a locally compact abelian group, then \( \Lambda(G) \) is abelian locally compact. This is a consequence of the Riemann-Lebesgue lemma.

Example 2.24. For \( G = \mathbb{C}^1 \), we have the Fourier coefficients
\[
\hat{\varphi}(n) = a_n = \int_{\mathbb{C}^1} \varphi(x)x^n dx
\]
and \( a_n \to 0 \) as \( n \to \infty \).

Example 2.25. Let \( G = \mathbb{R} \). Then
\[
\hat{\varphi}(\xi) = \int_{\mathbb{R}} \varphi(x) \exp(-2\pi i \xi x) dx
\]
is continuous, and \( \hat{\varphi}(\xi) \to 0 \) as \( \xi \to \infty \).

2.3.2. Fourier inversion theorem. Let \( \varphi : \mathbb{C}^1 \to \mathbb{C} \), then when does \( \varphi(x) = \sum_{n \in \mathbb{Z}} a_n x^{-n} \) converge. The goal is to generalize this.

Theorem 2.26. Fourier adjunction formula. Let \( f \in L^1(G) \) and \( g \in L^1(\Lambda(G)) \). Then
\[
\int_G f(x) \hat{g}(x) dx = \int_{\Lambda(G)} \hat{f}(\chi) g(\chi) d\chi = \int_G \int_G f(x) g(\chi(x))^{-1} dx d\chi.
\]

Remark 2.27. At this point, \( d\chi \) can be anything. However, we do have a canonical choice. That is, choose it so that we have Fourier inversion formula (to come).
Consider the space of functions $f : G \to \mathbb{C}$ and $\hat{f}$ are continuous and in $L^1$. Then we want $d\chi$ so that
$$\int_{\Lambda(G)} \hat{f}(\chi) d\chi = f(0)$$
for all $f$. This will follow from the adjunction formula, and the existence of a Dirac sequence. When can just apply $\hat{g} \to \delta_0$ and $g \to 1$ to the adjunction formula.

**Example 2.28.** For $G = \mathbb{R}$, let $b_\infty(x) = e^{-\pi x^2}$ then $\hat{b}_\infty(\xi) = e^{-\pi \xi^2}$. We can scale $x \mapsto b_\infty(tx)$. Then
$$\lim_{n \to \infty} \int_G f(x) \hat{g}_n(x) dx = f(0).$$

**Example 2.29.** For $F = \mathbb{Q}_p$, we can let $b_p(x) = 1_{\mathbb{Z}_p}$ then $\hat{b}_p(\xi) = 1_{\mathbb{Z}_p}$. The Dirac sequence is obtained by scaling the basis functions.

**Theorem 2.30.** (Pontryagin) $G \to \Lambda(\Lambda(G))$ by $x \mapsto (\chi \mapsto \chi(x))$ is an isomorphism of locally compact groups.

More properties:

1. Let $(\ell_x f)(y) = f(x^{-1}y)$ then $\hat{\ell_x f}(\chi) = \chi(x)^{-1} \hat{f}(x)$
2. There is a multiplication on $L^1(G, dx)$ given by convolution. For $f, g \in L^1(G, dx)$ then
   $$(f * g)(x) = \int_G f(y) g(y^{-1}x) dy.$$

   Then $f * g \in L^1(G, dx)$. Note, the map $y \mapsto f(y) g(y^{-1}x)$ is only integrable for almost all $x$ (not necessarily all). Additionally,
   $$\int_G (f * g)(x) = \int_G f(x) dx \int_G g(x) dx.$$

3. Fubini’s theorem implies that $\hat{f} * \hat{g}(\chi) = f(\chi) g(\chi)$.

**Remark 2.31.** That is, Fourier transform converts convolution product into pointwise multiplication.

We can rewrite $\ell_x f = \delta_x * f$.

2.3.3. **Topology on $\Lambda(G)$**. We gave it the compact open topology (so a sequence of function converges, if they uniformly converge on every compact subset).

**Theorem 2.32.** $\Lambda(G)$ with compact open topology is locally compact.

**Lemma 2.33.** (Riemann-Lebesgue) Suppose $f \in L^1(G)$, then $\hat{f}$ is continuous on $\Lambda(G)$ (it tends to $0$ as $\chi \to \infty$). $\hat{f}$ extends as a continuous function to the one point compactification $\Lambda(G) \cup \{\infty\}$, by setting $\hat{f}(\infty) = 0$. The topology on the compactification is that the neighbourhood of $\infty$ is the complement of compact subsets of $\Lambda(G)$.

**Proof.** For all $\epsilon > 0$, consider
$$K_\epsilon = \left\{ \chi \in \Lambda(G) : \left| \hat{f}(\chi) \right| \geq \epsilon \right\}$$
is compact. To prove this, use Ascoli theorem and the fact that translations $\ell_x$ is continuous on $L^1(G)$. □

Consider the map $\psi : G \to \Lambda(\Lambda(G))$ by $x \mapsto (\chi \mapsto \chi(x))$.

**Proposition 2.34.** Fourier adjunction formula. For $f \in L^1(G, dx)$ and $g \in L^1(\Lambda(G), d\chi)$ then
$$\int_G \hat{f}(\chi) g(\chi) d\chi = \int_G f(x) \hat{g}(\psi(x)) dx.$$

**Proof.** Application of Fubini. □

**Theorem 2.35.** Fourier inversion theorem. Let $G$ be a locally compact abelian group and $\Lambda(G)$ its Pontryagin dual. For all Haar measure $dx$ on $G$, there exists a unique Haar measure $d\chi$ on $\Lambda(G)$ satisfying:
• For all \( f \in L^1(G, dx) \) such that \( \hat{f} \in L^1(\Lambda(G), d\chi) \) then
\[
f(0) = \int_G \hat{f}(\chi) d\chi.
\]

In this case, both \( f \) and \( \hat{f} \) are continuous and vanishing at \( \infty \).

**Proof.** Use Dirac sequence. \( \square \)

**Theorem 2.36.** (Pontraygin) The map \( G \to \Lambda(\Lambda(G)) \) is an isomorphism.

**Proof.** Fourier adjunction and inversion. \( \square \)

**Proposition 2.37.** Abstract Poisson Summation Formula. Let \( G \) be a locally compact abelian group. Let \( H \subseteq G \) be a closed subgroup.
\[
0 \to H \to G \to G/H \to 0.
\]

Let
\[
H^\perp = \left\{ \chi \in \hat{G} : \chi|_H = 1 \right\}
\]

which is a closed subgroup of \( \hat{G} \) giving us
\[
0 \to H^\perp \to \hat{G} \to \hat{G}/H^\perp \to 0.
\]

Then \( \Lambda(H) = \hat{G}/H^\perp \) and \( \Lambda(H^\perp) = G/H \).

**Proof.** We compare the Fourier transforms. Let \( f \in L^1(G) \) and \( \varphi(x) = \int_H f(xh) dh \) which is a function of \( G/H \). This is well-defined almost everywhere by Fubini, and
\[
\varphi \in L^1(G/H, dg/dh).
\]

Then \( \varphi \in C(H^\perp) \) which is the restriction of \( \hat{f} \) to \( H^\perp \) almost everywhere. \( \square \)

**Example 2.38.** Let \( G = \mathbb{R} \) and \( H = \mathbb{Z} \). Let \( f \in L^1(G) \), then
\[
\varphi(x) = \sum_{n \in \mathbb{Z}} f(x + n)
\]
a function on \( \mathbb{R}/\mathbb{Z} \). Assume that everything converges, \( f, \hat{f} \in L^1 \), \( \varphi, \hat{\varphi} \) absolutely converges. Apply Fourier inversion to \( \varphi \) then
\[
\varphi(0) = \sum_{n \in \mathbb{Z}} f(n)
\]
and
\[
\int_{\mathbb{Z}} \hat{\varphi} = \sum_{n \in \mathbb{Z}} \hat{f}(n).
\]
The usual Poisson summation formula says
\[
\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).
\]

2.4. **Fourier theory on additive groups.** Let \( F \) be a local field (just \( \mathbb{R}, \mathbb{C}, \mathbb{Q}_p \) or \( F/\mathbb{Q}_p \) finite extension). What is \( \Lambda(F) \)? It is just \( F \).

We have a multiplication action of \( F^\times \) on \( F \), and so on \( \Lambda(F) \). Therefore, \( \Lambda(F) \) must be a \( F \)-vector space. If \( \Lambda(F) = 0 \) then \( \Lambda(\Lambda(F)) = 0 \) which is a contradiction. If \( \dim_F \Lambda(F) \geq 2 \), then \( \Lambda(F) = F \oplus H \). Then \( \Lambda(\Lambda(F)) = \Lambda(F) \oplus \Lambda(H) \) so the dimension is too great. Hence, \( \Lambda(F) = F \). Note that if \( F = \mathbb{Q} \), then \( F \) is not closed in \( \Lambda(F) \), so we can’t write this \( \Lambda(F) = F \oplus H \).

The Pontryagin duality theorem implies that \( \dim_F \Lambda(F) = 1 \). We will construct a preferred element. If \( F = \mathbb{R} \), \( \psi_1(x) = \exp(2\pi i x) \). If \( F = \mathbb{C} \), \( \psi_1^\mathbb{C}(x + iy) = \psi_1^\mathbb{R}(2x) \). When \( F = \mathbb{Q}_p \), for \( x \in \mathbb{Q}_p \) of the form \( x = a + b \) where \( a \in \mathbb{Z}_p \), \( b = m/p^n \) with \( 0 \leq m < p^n \). Then \( \psi_1^\mathbb{Q}_p(x) = \exp(-2\pi ib) \). Extend this to all of \( \mathbb{Q}_p \).
Then for all $x \in \mathbb{Q}$,

$$\prod_v \psi_1^F(x) = 1.$$ 

For $F/\mathbb{Q}_p$ finite extension, let

$$\psi_1^F(x) = \psi_1^{\mathbb{Q}_p}(\text{Tr}_{f/\mathbb{Q}_p}(x)).$$

Then for all number fields $K/\mathbb{Q}$, and $x \in K$,

$$\prod_v \psi_1^F(x) = 1.$$ 

With the choice of a square root of $-1$, we have constructed in a consistent way, additive characters $\psi_1^F$ for all local fields $F$.

The characters (non-unitary) of $\mathbb{R}$ and $\mathbb{Q}_p$ are very different. The character $\mathbb{R} \to \mathbb{R}_+$ given by $x \mapsto \exp(x)$ is non-unitary. On the other hand, all characters of $\mathbb{Q}_p$ are unitary. This is because $\mathbb{Q}_p$ is a union of compact subgroups and they must be mapped to $\mathbb{C}^1$.

Let $\mu$ be a Haar measure of $F$, and $\nu$ dual Haar measure on $\Lambda(F) = F$. For all $\alpha \in \mathbb{R}_+^+$, the dual Haar measure of $\alpha \mu$ is $\alpha^{-1} \nu$. We will now compute self-dual Haar measure on local fields.

For $F = \mathbb{R}$, this is just the usual Lebesgue measure. For $F = \mathbb{C}$, it’s the measure $2dxdy$. If $F = \mathbb{Q}_p$, it’s the one where $\text{vol}(Z_p) = 1$ (so $1_{\mathbb{Z}_p}$ is self-dual). For $F/\mathbb{Q}_p$,

$$1_{\mathbb{O}_F} = \{x \in F : \text{for all } a \in \mathbb{O}_F, \text{Tr}(ax) \in \mathbb{Z}_p\}.$$ 

Therefore, the dual of $1_{\mathbb{O}_F}$ is the volume of $\mathbb{O}_F$. The self-dual Haar measure is then where $\text{vol}(\mathbb{O}_F) = |\text{Nm}(\mathbb{O}_F^\perp)|^{-1}$.

$$\psi_1^F : F \to \mathbb{C}^1 \text{ identification } F \cong \Lambda(F) \text{ and so gives a canonical self-dual measure.}$$

Fix $d_F x$ on $F$, giving $d_{\mathbb{A}} x$ on $\mathbb{A}$.

Since

$$\int 1_{\mathbb{O}_v} dx = |\text{Nm}(\mathbb{O}_v^\perp)|^{-1/2}$$

and $\varphi = \otimes \varphi_v$ with $\varphi_v = 1_{\mathbb{O}_v}$ for almost all $v$,

$$\int \varphi = \prod \int \varphi_v$$

and the second part is 1 for almost all $v$.

For $b_v = 1_{\mathbb{O}_v}$, $b_v = \text{vol}(\mathbb{O}_v) \cdot 1_{\mathbb{O}_v}^\perp$ and so

$$\int b_v dx = \text{vol}(\mathbb{O}_v) \text{ vol}(\mathbb{O}_v^\perp) = |\text{vol}\mathbb{O}_v|^2 |\text{Nm}(\mathbb{O}_v^\perp)| = 1.$$

**Remark 2.39.** Since $\mathbb{A}/K$ is compact, for $a \in K^\times$ and $\prod_v |a|_v = 1$, for $a \in F^\times$, $d(ax) = |a| dx$. Let

$$B_c = \{x \in \mathbb{A} : |x_v| \leq c_v\}$$

where $c = (c_v)$ is such that $c_v = 1$ for almost all $v$. $B_c$ is a fundamental domain of $K \subseteq \mathbb{A}$.

Let $f = 1_{B_c}$ then $\varphi(x) = \sum_{a \in k} f(x + a) \in L^1(\mathbb{A}/K)$ and $\varphi(x) = 1$ for almost all $x$.

$$\text{vol}(B_c, dx) = \text{vol}(\mathbb{A}/K, dy)$$

and

$$\int_{\mathbb{A}} f(x) dx = \int_{\mathbb{A}/K} \varphi(y) dy.$$

For all $a \in K^\times$, if $B_C$ is a fundamental domain, so is $B_{|a| |c_v|}$. Therefore,

$$\text{vol}(B_{c_0}) = \text{vol}(B_{|a| |c_v|}) = \text{vol}(\mathbb{A}/K).$$

**Proposition 2.40.** If $dx$ is the self-dual Haar measure on $\mathbb{A}$ with respect to $\psi_1^\mathbb{A}$, then $\text{vol}(\mathbb{A}/K) = 1$.

**Proof.** Explicitly show that $\text{vol}(\mathbb{O}_v) = |\text{discriminant}|^{-1/2}$. 


Another proof is using the Poisson summation formula.

\[ 0 \to K \to A \to A/K \to 0. \]

Here, \( K \) is discrete and \( A/K \) is compact. Using \( \psi_1 \), we get

\[ 0 \to K^\perp = K \to A(\hat{\Lambda}) = A \to A/k \to 0. \]

Let \( f \in L^1(A) \) assume that \( \hat{f} \in L^1(\hat{A}) \) (which implies they are continuous). Then \( \varphi \in L^1(A/K) \) maps to \( \hat{\varphi} = \hat{f} \mid_K \) by Fubini.

Assume \( \hat{\varphi} \in L^1(K) = L^1(\Lambda(\hat{A}/K)) \) (that is, \( \sum_{a \in K} |\hat{f}(a)| \) converges). Fourier inversion says that

\[ \varphi(0) = \int_K \hat{\varphi}(a) da. \]

Here, \( da \) is the Haar measure on \( K \), dual to \( dy \) on \( A/K \). This is

\[ = \text{vol}(A/K, dy) \sum_{a \in K} \hat{\varphi}(a). \]

Since \( \varphi \) and \( \hat{\varphi} \) are continuous, and \( \varphi(x) = \sum_{a \in K} f(x + a) \), we see that \( \varphi(0) = \sum_{a \in K} f(a) \). Exchange \( f \) and \( \hat{f} \) then \( \text{vol}(A/K, dy) \).

\[ \square \]

**Corollary 2.41.** If \( f \in L^1(A) \) and \( \hat{f} \in L^1(\hat{A}) \) then \( \sum_{a \in K} f(a) = \sum_{a \in K} \hat{f}(a) \) as long as both sums converge absolutely.

2.5. **Schwartz Functions.** Let \( V \) be a finite dimensional \( \mathbb{R} \)-vector space. We say \( \varphi : V \to \mathbb{C} \) is a rapid decaying function if for all polynomial functions \( p : V \to \mathbb{C} \), we have \( |p\varphi| \) is bounded. Then it must be that \( \varphi \in L^1(V) \). Let

\[ \mathcal{S}(V) = \{ \varphi \text{ measurable function on } V \text{ such that } \varphi \text{ and } \hat{\varphi} \text{ are of rapid decay} \} . \]

**Proposition 2.42.** \( \varphi \in \mathcal{S}(V) \) iff \( \varphi \) is smooth and for all polynomials \( p \) and \( n \), \( |p\varphi^{(n)}| \) is bounded.

**Proof.** Let \( V = \mathbb{R} \). Then

\[ \varphi(x) = \int \hat{\varphi}(y)e(xy)dy. \]

Since \( \hat{\varphi} \) is of rapid decay, we can differentiate under the integral sign. This implies that \( \varphi \) is smooth, and similarly for \( \hat{\varphi} \).

\[ \square \]

Recall that for \( \mathbb{R} \), \( b_\alpha = e^{-\pi \alpha^2} \) is of rapid decay, and is its own Fourier transform. It is a Schwartz function.

Now, for non-archimedean fields \( F \). Then \( V \) is totally disconnected locally compact abelian groups with \( V = \Lambda(V) \). Let \( \varphi \) be a continuous function on \( V \) with compact support.

\[ \mathcal{S}(V) = \{ \varphi \text{ continuous function on } V \text{ with compact support, and same with } \hat{\varphi} \} . \]

**Proposition 2.43.** \( \varphi \in \mathcal{S}(V) \) iff \( \varphi \) is a linear combination of characteristic functions of compact open subsets.

**Proof.** Since \( V \) is totally disconnected, we have a system of neighbourhoods of 0 consisting of compact open subgroups \( K_\alpha \) with \( \cap K_\alpha = \{0\} \). Then \( \cup K_\alpha^\perp = V. \)

\( \varphi \) is invariant under \( K_\alpha \) for some \( \alpha \), and

\[ \varphi(x) = \int_V \hat{\varphi}(y)e(xy)dy. \]

If \( \text{supp}(\hat{\varphi}) \subseteq K_\alpha^\perp \) then this is equal to

\[ \int_{K_\alpha^\perp} \hat{\varphi}(y)e(xy)dy. \]

For \( y \in K_\alpha^\perp \), \( x \mapsto e(xy) \) is \( K_\alpha \)-invariant.

\[ \square \]
We have
\[ \mathcal{S}(\mathbb{A}_F) = \mathcal{S}(\mathbb{A}_{F_{\text{fin}}}) \otimes \mathcal{S}(F) \]
where \( \varphi \in \mathcal{S}(\mathbb{A}_{F_{\text{fin}}}) \) and \( \varphi \in \mathcal{S}(F) \).

For \( \varphi \in \mathcal{S}(A) \),
\[ \sum_{a \in K} \varphi(a) \]
is absolutely convergent and so
\[ \sum_{a \in K} \varphi(a) = \sum_{a \in K} \hat{\varphi}(a). \]

To prove absolute convergence, we reduce to the usual case. Since \( \text{supp}(\varphi) \subseteq K \) for some compact open subgroup of \( \mathbb{A}_{F_{\text{fin}}} \),
\[ \sum_{a \in K} |\varphi(a)| \leq c \sum_{a \in K \cap K} \varphi_{\text{c}}(a). \]

The \( K \cap K \) is a lattice in \( K \) (think \( \mathbb{Q} \cap \hat{\mathbb{Z}} = \mathbb{Z} \subseteq \mathbb{R} \)). Therefore, this converges.

2.6. Tempered Distribution on \( F \). Define
\[ \mathcal{S}^*(F) = \{ \text{continuous linear form on } \mathcal{S}(F) \}. \]

When \( F \) is \( p \)-adic, we use the discrete topology. Otherwise, we use the norm topology.

**Example 2.44.** For \( x \in F \), let \( \delta_x(\varphi) = \varphi(x) \). This is the Dirac function.

**Example 2.45.** Let \( f \) be a continuous function on \( K \). Then \( f \in \mathcal{S}^*(F) \) as the map \( \varphi \mapsto \int \varphi(x)f(x)dx \).

**Example 2.46.** For \( a \in \mathcal{S}^*(F) \) get \( \hat{a} \in \mathcal{S}^*(F) \) defined by \( \hat{a}(\varphi) = a(\varphi) \).

**Example 2.47.** \( \hat{\delta}_x = 1_F \) and \( \hat{\delta}_\psi = \psi^{-1} \) (mapping \( \chi \mapsto \chi(x)^{-1} \)).

Suppose \( \chi : F^\times \to \mathbb{C}^\times \) is a character. What is \( \hat{\chi} \)? If \( \chi(x) = |x|^s \) with \( \text{Re}(s) > 0 \), then \( x \) can be extended continuously to 0 and \( \chi \in \mathcal{S}^*(F) \) (really, just need \( \text{Re}(s) > -1 \)).

We want to extend almost all \( \chi \) to a tempered distribution. We will get a formula
\[ \hat{\chi}^{-1} = \gamma(\chi) \chi^{-1} \]
where \( \gamma(\chi) \) is some constant.

3. Module in Local Fields

Let \( F \) be a local field, with invariant measure \( \mu \). For \( f \in C_c(F) \), we let \( \mu(f) = \int_X f(x)\mu(x) \) (here, \( X \) is the support of \( f \)). We have an action of \( F^\times \) on \( F \), on functions, and on measures. For \( t \in F^\times \), \( x \in F \), \( \varphi \in C_c(F) \) and \( \mu \) a measure, we act by
\[ (t, x) \mapsto tx, \varphi_t(x) = \varphi(t^{-1}x) \text{ and } \mu_t(\varphi) = \mu(\varphi_{t^{-1}}). \]

Then
\[ \langle \mu_t, \varphi \rangle = \langle \mu, \varphi \rangle. \]

Let \( \mu \) be the additive invariant measure on \( F \) (for instance, the self-dual measure depending on \( \psi : F \to \mathbb{C}^\times \)). \( \mu_t \) is then also invariant measure, with
\[ \mu_t = |t|^{-1} \mu \]
and here, \( |t|^{-1} \) is called the module (modulus?). This is because
\[ \int \varphi(x)\mu_t(x) = \int \varphi_{t^{-1}}(x)\mu(x) = \int \varphi(tx)\mu(x) \]
then use change of variable with \( y = tx \).

Use the usual absolute value over \( \mathbb{R} \), and the absolute value \( |x + iy| = x^2 + y^2 \) on \( \mathbb{C} \). For \( F \) a local non-archimedean field, let \( |a|_F = q^{-\nu(a)} \). Define the absolute value on \( \mathbb{A}_F^\times \) by

\[
(x_v) \mapsto \prod |x_v|_{K_v}.
\]

Now, how do we normalize the multiplicative measure. Let \( \omega^1 : F^\times \to \mathbb{C}^\times \) be given by \( x \mapsto |x| \). Consider the measure \( \mu \) on \( C_c(F^\times) \) be the same as the measure on \( C_c(F) \) via extension by 0.

**Claim 3.1.** \( \omega^{-1} \mu \) is an invariant measure on \( F^\times \) (the measure \( \frac{\mu(x)}{|x|} \)).

For \( F = \mathbb{Q}_p^\times \), vol \((\mathbb{Z}_p^\times, \omega^{-1} \mu) = p^{-1} = 1 - \frac{1}{p} \) and \( \text{vol}(\mathbb{Z}_p, \mu) = 1 \).

For \( F \) non-archimedean local field,

\[
\text{vol}(O_F^\times, \mu) = \text{vol}(O_F^\times, \omega^{-1} \mu) = (1 - q^{-1}) \text{disc}_{F/\mathbb{Q}_p}^{-1/2} = (1 - q^{-1}) \text{vol}(O_F, \mu).
\]

We know \( \mathbb{A}_F^\times = \mathbb{A}_F^\times \times K_R^\times \). The first is totally disconnected, and the latter is \((\mathbb{R}^\times)^1 \times (\mathbb{C}^\times)^2 \). Take the invariant measure to be \( \mu^\times = \mu_{\mathbb{A}^\times} \times \mu_{K_R^\times} \). The invariant measure on \( \mathbb{A}_F^\times \) is determined by

\[
\text{vol} \left( \prod O_v, \mu \right) > 0,
\]

because it is a compact open subgroup of \( \mathbb{A}_F^\times \). If we choose \( \mu = \prod |\omega|_v^{-1} \mu_v \) and when \( K = \mathbb{Q} \), we have

\[
\text{vol} = \prod (1 - p^{-1}) = 0.
\]

This means that we need to normalize so that \( \text{vol}(O_v^\times, \mu^\times) = 1 \). That is, let \( \mu_v^\times = (1 - q^{-1})^{-1} \text{disc}_{F/\mathbb{Q}_p}^{-1/2} \omega^{-1} \mu_v \).

Suppose \( F \) is archimedean, and take \( \mu^\times = \omega^{-1} \mu \). We have the exact sequence

\[
0 \to \mathbb{A}^1 \to \mathbb{A}_F^\times \to \mathbb{R}_+^\times \to 0
\]

which has a splitting, by choice of archimedean places. This means \( \mathbb{A}_F^\times = \mathbb{A}_F^1 \times \mathbb{R}_+^\times \), so let \( \mu_{\mathbb{A}^1} = \mu_{\mathbb{A}^1} \times \mu_{\mathbb{R}_+^\times} \) to give a measure on \( \mathbb{A}^1 \). Recall that \( \mathbb{A}_F^1/K^\times \) is compact.

**Proposition 3.2.** \( \text{vol}(\mathbb{A}^1/K^\times, \mu_{\mathbb{A}^1}) = \frac{2^s(2\pi)^2 R}{\sqrt{|d|}} \) where \( h \) is the class number, \( R \) regulator, \( d \) is the discriminant and \( w \) is the number of unit roots.

**Proof.** Hint:

\[
0 \to K^\times/R^\times \to \mathbb{A}^1/K^\times \prod O_v^\times \to \mathbb{C} \to 0.
\]

Recall

\[
\Omega(F^\times) = \{ \chi : F^\times \to \mathbb{C}^\times \text{ not necessarily unitary} \}
\]

and it has a distinct character \( \omega^1 \) (the module character). For all \( s \in \mathbb{C} \), let \( \omega^s(x) = |x|^s \). This defines a homomorphism \( \mathbb{C} \to \Omega(F^\times) \), and this is injective if \( F = \mathbb{R} \) or \( \mathbb{C} \).

For \( F \) non-archimedean, \( \omega^1 : F^\times \to \mathbb{R}_+^\times \) factors through \( q^2 \). This means that we have

\[
0 \to \frac{2\pi i}{\log q} \to \mathbb{C} \to \Omega(F^\times) \to \Omega(\ker \omega^1) \to 0
\]

The kernel \( \ker(\omega^1) \) is a compact subgroup of \( F^\times \). When \( F = \mathbb{R} \), this is \( \pm 1 \), when \( F = \mathbb{C} \), this is \( \mathbb{C}^1 \) and for non-archimedean, it is \( O_F^\times \).

Then, \( \Omega(F^\times) \) is an extension of the discrete group \( \Lambda(\ker \omega^1) = \Omega(\ker \omega^1) \) by \( \mathbb{C} \) over the quotient \( \mathbb{C} \) by \( \mathbb{Z} \).

For \( \mathbb{R}_+^\times = \mathbb{R}_+^\times \times \{ \pm 1 \} \) and \( \Omega(\mathbb{R}_+^\times) = \mathbb{C} \) and \( \Omega(\pm 1) = \pm 1 \). This means that \( \Omega(\mathbb{R}^\times) = \mathbb{C} \times \mathbb{C} \).

For \( \mathbb{C}^\times = \mathbb{R}_+^\times \times \mathbb{C} \), \( \Omega(\mathbb{C}^\times) = \Omega(\mathbb{R}_+^\times) \times \Omega(\mathbb{C}^1) = \mathbb{C} \times \mathbb{C} \).

For \( F^\times = O_F^\times \times \mathbb{Z} \), \( \Omega(F^\times) = \Omega(O_F^\times) \times \Omega(\mathbb{Z}) = \Omega(O_F^\times) \times \mathbb{C}/\mathbb{Z} \) (which are \( \frac{2\pi i}{\log q} \)).
Let $\chi : F^x \to \mathbb{C}^x$ then we can define $|x| : F^x \to \mathbb{R}^x_+$. This will satisfy $|\chi|(x) = |x|^s$ for some $s \in \mathbb{C}$, because $|\chi|_{|F_p^x|} = 1$. We define $Re(\chi) = Re(s)$. Then define an exponent map $\Omega(F^x) \to \mathbb{R}$ by $x \mapsto Re(\chi)$.

This can be applied to $\mathbb{A}^x/K^x \rtimes \mathbb{R}^x_+$. ker$(\omega_k) = \mathbb{A}^1/K^x$ some gigantic compact group.

3.1. Mellin transform over $\mathbb{R}^x_+$. Let $\varphi : \mathbb{R}^x_+ \to \mathbb{C}$. We know $\Omega(\mathbb{R}^x_+) = \mathbb{C}$. Define

$$M_\varphi(s) = \int_{\mathbb{R}^x_+} \varphi(x) x^s \mu^x(x)$$

for all $s \in \mathbb{C}$.

Recall

$$\mathcal{S}(\mathbb{R}^x_+) = \{ \varphi : \mathbb{R}^x_+ \to \mathbb{C} \text{ of rapid decay at } 0 \text{ and } \infty \}$$

That is, $|x^\sigma \varphi^{(n)}(x)|$ bounded for all $\sigma$, and so if $\sigma > 0$, bounded as $x \to \infty$; and if $\sigma < 0$, as $x \to 0$.

**Theorem 3.3.** For $\varphi \in \mathcal{S}(\mathbb{R}^x_+)$, $M_\varphi(s)$ is absolutely converge for all $s \in \mathbb{C}$ and defines a holomorphic function $g = M_\varphi$ of variable $s$. They lie in

$$Z(\mathbb{C}) = \{ g : \mathbb{C} \to \mathbb{C} \text{ hol.}, \text{and } \forall P(s) \text{ polynomial, } p(s)g(s) \text{ is bounded in every vertical strip} \}.$$ 

If $g \in Z(\mathbb{C})$ then

$$\varphi = M_g^{-1}(x) = \frac{1}{2\pi i} \int_{\sigma+i\mathbb{R}} g(x) x^{-s} ds.$$

**Proof.** We have

$$\int_\sigma^\alpha \varphi(x) x^s \mu^x(x) = \int_{x<1} + \int_{x>1}. $$

For the $x<1$ portion, we see that $|\varphi(x)x^s| < cx^2$ and for $x > 1$, $|\varphi(x)x^s| < cx^{-2}$. Therefore, we have absolute convergence.

We know that $\partial_x x^s = \log(x)x^s$.

$$\int_{x<1} \varphi \log(x)x^s + \int_{x>1} \varphi(x) \log(x)x^s$$

absolutely converges. Since $M\varphi$ has complex derivatives, it is holomorphic.

Write $g = M_\varphi$. Consider $\mathbb{R} \to \mathbb{R}^x_+$ by $y \mapsto x = e^y$.

$$M(\varphi)(\sigma + it) = \int_{\mathbb{R}} \varphi(e^y)e^{\sigma y} e^{iyt} \mu(y).$$

The $e^{iyt}$ is an additive character, and so $M\varphi(\sigma + it)$ viewed as function of $t$ is the Fourier transform of $\varphi(e^y)e^{\sigma y}$. The function $y \mapsto \varphi(e^y)e^{\sigma y}$ is in $L^1$ and so $M\varphi(\sigma + it)$ as function of $t$ is continuous and bounded (by the $L^1$-norm). This means it is bounded on vertical strips.

For boundedness, use invariant differential operator $xdx$ (with respect to multiplication).

$$M(x\delta_x \varphi) = \int_\alpha^\alpha \delta_x \varphi(x) x^{s+1} dx = -sM_\varphi(s)$$

which shows that $|sM_\varphi(s)|$ is bounded.

Now,

$$M^{-1}g(x) = \frac{1}{2\pi i} \int_{-\alpha}^\alpha g(\sigma + it)x^{-(\sigma + it)} \mu(t)$$

is again a Fourier transform. Since $P(s)g(s)$ is bounded on vertical strips, $g(\sigma + it)$ in $L^1$ (for fixed $\sigma$). Apply Fourier inverse to get $M^{-1}g = \varphi$.

Need to check $M^{-1}g$ is smooth, rapid decay and things. Smooth follows from differentiating inside the integral. For rapid decay, it’s from the $x^{-\sigma}$ factor.

$$\varphi(x)x^\sigma = \frac{1}{2\pi i} \int_{-\alpha}^\alpha g(\sigma + it)x^{-it} \mu(t)$$

is bounded, because $g$ is $L^1$. Identify $\mathbb{R} \to \mathbb{R}^x_+$ by $y \mapsto e^{iy}$. Then $\varphi(x)x^\sigma = \varphi(e^y)e^{\sigma y}$ gives Fourier transform $g(\sigma + it)$.
Let $\mathcal{S}(\mathbb{R}^+)$ be the set of $\varphi: \mathbb{R}^+ \to \mathbb{C}$ smooth, such that

- for all $\sigma > 0$, $|x^\sigma \varphi^{(n)}(x)|$ is bounded (rapid decay as $x \to \infty$)
- $\varphi$ has a Taylor expansion at 0, $\sum_{n \geq 0} a_n x^n$ formal series (not analytic). That is, for all $n$, $\varphi(x) = \sum_{i=0}^n a_i x^i + O(x^n)$.

**Theorem 3.4.** Let $Z_+(\mathbb{C})$ be meromorphic functions $g$ of variable $s \in \mathbb{C}$, with at most simple poles at 0, $-1, -2, ..., \sigma$ such that for all polynomial $P(s)$, $P(s)g(s)$ is bounded on every vertical strip often deleting a small disk centered at the poles.

If $\varphi \in \mathcal{S}(\mathbb{R}^+)$ then $g = M_\varphi \in Z_+(\mathbb{C})$. If $g \in Z_+(\mathbb{C})$, then $\varphi = N_g \in \mathcal{S}(\mathbb{R}^+)$ with $\sigma > 0$. The $M$ and $N$ are inverses of each other.

The Taylor coefficients $a_0, a_1, ..., \varphi$ at 0 are exactly the residues of $g$ at $0, -1, -2, ...$.

**Proof.** The fact that $|\varphi^{(n)}(x)x^\sigma|$ is bounded for $\sigma > 0$, implies that

$$M_\varphi(s) = \int_{\mathbb{R}^+} \varphi(x)x^s \mu^x(x)$$

is integrable, for $\sigma = \text{Re}(s) > 0$.

Meromorphic continuation of $M_\varphi$, to get for $\text{Re}(s) > 1$

$$M(\partial_x \varphi)(s) = \int_0^\infty \partial_x \varphi(x)x^{s-1} dx = -(s-1) \int_0^\infty \varphi(x)x^{s-2} dx = -(s-1)M_\varphi(s-1).$$

We take this to be the formula,

$$M_\varphi(s-1) = -(s-1)^{-1}M(\partial_x \varphi)(s)$$

giving $M_\varphi$ defined for $\text{Re}(s) > 1$ with a pole at 0. We can repeat this, to get pole at 0, $-1, -2, ...$.

For $p(s)g(s)$ bounded on vertical strips, use invariant differential operator $x\partial_x$ to get

$$M(x\partial_x \varphi)(s) = -sM_\varphi(s).$$

Now,

$$s^{-1}(s-1)^{-1}...(s-n)^{-1}$$

is bounded on $\mathbb{C} - (D_0 \cup D_1 \cup ... \cup D_n)$ (the vertical strips). This proves the boundedness of $p(s)g(s)$.

Suppose $g: \mathbb{C} \to \mathbb{C}$ meromorphic with poles at $0, -1, ...$

$$\frac{1}{2\pi i} \int_{\sigma+i\mathbb{R}} g(s)x^{-s} ds.$$ 

If $-(n-1) < \sigma < -n$, this integrals is just the sum of the residues

$$\varphi(x) = \sum_{i=0}^{-n} \text{res}_i \left(g(s)x^{-s} ds\right) + O(x^{-\sigma}) = \sum_{i=0}^{-n} x^i \text{res}_i g(s) ds + O(x^{-\sigma}).$$

They become the Taylor coefficients as required.

**Proposition 3.5.** $\mathcal{S}(\mathbb{R}^+)$ is an algebra with respect to convolution, that is

$$(\varphi_1 * \varphi_2)(x) = \int_{\mathbb{R}^+} \varphi_1(y) \varphi_2(y^{-1}x) \mu^x(y).$$

Additionally,

$$M_{\varphi_1 * \varphi_2}(s) = M_{\varphi_1}(s)M_{\varphi_2}(s).$$

**Remark 3.6.** It won't hold of $\mathcal{S}(\mathbb{R}^+_e)$, because when we take Mellin transform, we pick up simple poles. The product will then get double poles.

A good reference is Iguasa “higher form...”

However, $\mathcal{S}(\mathbb{R}^+_e)$ is a module over $\mathcal{S}(\mathbb{R}^+_e)$ with generator $e^{-x}$.

$$M(e^{-x})(s) = \int_0^\infty e^{-x}x^s \frac{dx}{x} = \Gamma(s)$$
for $\Re(s) > 0$. Recall that we prove meromorphic continuation of $\Gamma(s)$ exactly by applying the differential operator, and get $\Gamma(s + 1) = s\Gamma(s)$.

$\Gamma$ has simple pole at 0, −1, −2, ... and is a generator of $Z_+(\mathbb{C})$. $\Gamma$ is non-vanishing away from 0, −1, ...

$$\Gamma(s)\Gamma(1-s) = \frac{2\pi i}{e^{\pi i s} - e^{-\pi i s}}$$

has simple pole in $\mathbb{Z}$ and non-vanishing on $\mathbb{C} - \mathbb{Z}$.

For all $\varphi \in \mathcal{S}_+(\mathbb{R})$, $\varphi = e^{-x} \ast \varphi_0$ for some $\varphi_0 \in \mathcal{S}(\mathbb{R}_+^\times)$.

### 3.2. Mellin Transform on $\mathbb{R}^\times$.

We can write $\mathbb{R}^\times = \mathbb{R}_+^\times \times \{\pm 1\}$. We know $\Omega(\mathbb{R}^\times) = \mathbb{C} \amalg \mathbb{C}$, where the first $s \in \mathbb{C}$ maps to $(x \mapsto |x|')$ the other maps $x \mapsto \text{sgn}(x)|x|^\sigma$.

For $\varphi \in \mathcal{S}(\mathbb{R})$, $\chi \in \Omega(\mathbb{R}^\times)$, let

$$M_\varphi(\chi) = \int_{\mathbb{R}^\times} \varphi(x)\chi(x)\mu^\times(x)$$

which converges for all $\sigma = \Re(s) > 0$. Have meromorphic continuation of $\Omega(\mathbb{R}^\times)$ with simple poles at 0, −2, −4, ... in $\Omega_+(\mathbb{R}^\times)$ and −1, −3, ... in $\Omega_-(\mathbb{R}^\times)$.

We write $\varphi = \varphi_+ + \varphi_-$, the even and odd Schwartz functions. Then

$$M_{\varphi_+}|_{\Omega_-} = 0 = M_{\varphi_-}|_{\Omega_+}.$$

$\varphi_+$ only has even Taylor coefficients, and $\varphi_-$ only has odd ones.

Let $\mathcal{L}(\Omega(\mathbb{R}^\times))$ be the set of meromorphic functions on $\Omega(\mathbb{R}^\times)$ with simple poles 0, −2, ... on $\Omega_+$ and −1, −3, ... on $\Omega_-(\mathbb{R})$. Also, $p(s)g(s)$ bounded on vertical strips, as before.

Our $L$-function is a generator of $\mathcal{L}(\Omega(\mathbb{R}^\times))$ (with some choice), as a module on holomorphic functions with boundedness condition on vertical strips.

Recall we have a basic function $b_{\infty}(x) = e^{-\pi x^2}$ which is self-dual under Fourier transform, and is an even function. Then

$$M_{\varphi_{\infty}}|_{\Omega_-} = 0 \text{ and } M_{b(\omega^s)} = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)$$

This latter part is what you multiply with $\zeta(s)$ to get functional equation!

**Exercise 3.7.** Work out Mellin transform on $\mathbb{C}^\times$.

### 3.3. Mellin Transform on non-archimedean local fields.

Let $F$ be a non-archimedean local field.

$$\mathcal{S}(F) = C_c^\infty(F) \supset C_c^\infty(F^\times).$$

This is similar for $\mathcal{S}(\mathbb{R}) \supset \mathcal{S}(\mathbb{R}^\times)$, but this time, we just have codimension 1 (value at 0).

$$\Omega(F^\times) = \{\chi : F^\times \to \mathbb{C}^\times\}$$

and

$$M_\varphi(\chi) = \int_{F^\times} \varphi(x)\chi(x)\mu^\times(x)$$

where $\mu^\times(\mathcal{O}_F^\times) = 1$.

$$\{M_\varphi : \varphi \in C_c^\infty(F^\times)\} = \mathcal{L}(\Omega(F)) \text{ and } \{M_\varphi : \varphi \in C_c^\infty(F)\} = \mathcal{L}_+(\Omega(F)).$$

We want to study these two spaces, and eventually discovering that $L(\chi)$ is a generator of $\mathcal{L}_+(\Omega(F))$. Recall that non-canonically, $F^\times = \mathcal{O}_F^\times \times \omega^\mathbb{Z}$ ($\omega$ uniformizer). This means that

$$\Omega(F^\times) = \Omega(\mathcal{O}_F^\times) \times \Omega(\omega^\mathbb{Z}) = \Lambda(\mathcal{O}_F^\times) \times \Omega(\omega^\mathbb{Z}).$$
Additionally, \[ \Lambda \left( \mathcal{O}_F^\times \right) = \lim_{\omega \to \infty} \left( \mathcal{O}_F / \omega^n \mathcal{O}_F \right)^\times \]
and is a union of finite subgroups. \( \Omega(\omega)^c \subset \mathbb{C}^\times \) by \( t \in \mathbb{C}^\times \) mapping to \( \{ \chi_t(\omega) = t \} \). \( \chi_t : F^\times \to \mathbb{C}^\times \) is trivial on \( \mathcal{O}_F^\times \), \( \chi_t(x) = |x|^s = \omega^s \) and so \( t = q^{-s} \). Hence, \( \Omega(F^\times) \) is infinite disjoint union of circles of \( \mathbb{C}^\times \).

**Definition 3.8.** \( g : \Omega(F^\times) \to \mathbb{C} \) is said to be polynomial of \( t \) if

- \( g \) vanishes on all but finitely many components
- on each component, \( g \) is a polynomial of \( t \)

**Proposition 3.9.** For all \( \varphi \in \mathcal{C}_c^\infty(F^\times) \), the Mellin integral \( M_\varphi(\chi) \) is convergent for all \( \chi \in \Omega(F^\times) \), and \( M_\varphi \) is a polynomial of variable \( t \). Conversely, if \( g \) is a polynomial on \( \Omega(F^\times) \), then

\[ \varphi(x) = N g(x) = \frac{1}{2\pi i} \int_{|t|=q^{-s}} g(t) \chi_t(x)^{-1} dt \]
(the integral being independent of \( \sigma \)) defines \( \varphi \in \mathcal{C}_c^\infty(F^\times) \).

**Proof.** Suppose \( \varphi \in \mathcal{C}_c^\infty(F^\times) \). For \( x \in F^\times \), there is a compact open subgroup \( K_x \) of \( F^\times \) such that \( \varphi \) is constant on \( xK_x \). \( \text{Supp}(F) \) is then a finite disjoint union of \( xK_x \) as above.

\( K = \cap_{\text{finite intersection}} K_x \)
is a compact open subgroup of \( F^\times \). \( \varphi \) is then a \( K \)-invariant. For all \( x \), \( \varphi \) is constant on \( xK \), unless \( \chi \big|_K = 1 \),

\[ \int_{F^\times} \varphi(x) \chi(x) \mu^\times(x) = 0. \]
This is because on a coset \( xK \), \( \int_K \chi(x) \mu^\times(x) = 0 \). Therefore, \( M_\varphi \) vanishes away from \( \Omega(\mathcal{O}_F^\times/K) \times \mathbb{C}^\times \).

\( \text{Supp}(\varphi) \) is compact, there exists \( m, n \in \mathbb{Z} \) such that

\[ \text{Supp}(\varphi) \subseteq \{ x \in F^\times : m \leq \text{val}(x) \leq n \} \].
Then \( \varphi = \sum_{i=m}^{n} \varphi_i \) where \( \text{Supp}(\varphi_i) = t^i \mathcal{O}_F^\times \), and \( \varphi_i(t^i x) = \sum_{\chi \in \Omega(\mathcal{O}_F^\times/K)} \varphi^\chi_i \) and

\[ \varphi^\chi_i(t^i x) = \chi(x) \varphi^\chi_i(t^i). \]
For all \( \chi \in \Omega(\mathcal{O}_F^\times/K) \), \( M_\varphi \big|_{\{ \chi \} \times \mathbb{C}^\times} \) is a linear combination of \( t^m, t^{m-1}, ..., t^n \) is a polynomial of \( t \).

For the inverse Mellin transform, it’s just Fourier series. \( \square \)

**Proposition 3.10.** For all \( \varphi \in \mathcal{C}_c^\infty(F) \), the Mellin transform is convergent for all \( x \in \Omega(F^\times) \) for \( |t| < 1 \), and holomorphically continued on all component \( \Omega_\chi = \{ \chi \} \times \mathbb{C}^\times \). For \( \neq 1 \), and meromorphically continued on the component \( \Omega_1 = \{ 1 \} \times \mathbb{C}^\times \) with a simple complex pole at \( t = 1 \). More precisely,

\[ M_\varphi \in \mathcal{L}_+(\Omega(F^\times)) = \{ g : \Omega(F^\times) \to \mathbb{C} \text{ rational function of } t \text{ such that } gL^{-1}(\chi) \text{ is a polynomial of } t \}. \]

Here,

\[ L(\chi) = \begin{cases} 1-t & \text{on the neutral component } \mathbb{C}^\times, \\ 1 & \text{everywhere else} \end{cases} \]

The inverse Mellin transform is given by

\[ Ng(x) = \frac{1}{2\pi i} \int_{|t|=q^{-s}} g(t) \chi_t(x)^{-1} dt \]
for \( \sigma > 0 \).
Proof. Compute the Mellin transform of the basis function \( b_F(x) = 1_{\mathcal{O}_F} \). This is \( \mathcal{O}_F^\times \)-invariant, \( M_{b_F} \) vanishes off the neutral component of \( \Omega(F^\times) \). On the neutral component \( \Omega_0(F^\times) = \mathbb{C}^\times \), \( M_{b_F}(t) \) is absolutely convergent for \( |t| < 1 \) because
\[
M_{b_F}(t) = \sum_{i=0}^{\infty} t^i = (1 - t)^{-1}
\]
which we can meromorphically continue.

For \( \varphi \in C_c^\infty(F) \), \( \varphi = \varphi(0)b_F + \varphi' \) where \( \varphi' \in C_c^\infty(F^\times) \). \( M_{\varphi} \) is polynomial function on \( \Omega(F^\times) \), \( M_{\varphi}(t) = (1 - t)^{-1} \) on \( \Omega_0(F^\times) \) and 0 elsewhere. This means the \( L \) function is \( (1 - t)^{-1} \) on the neutral component and 1 else. \( \square \)

4. Eigendistribution

\( F \) local field.

- \( F^\times \) acts on \( F \) by \( (t, x) \mapsto tx \).
- \( F^\times \) acts on \( \mathcal{S}(F) \) by \( \varphi_t(x) = \varphi(t^{-1}x) \).
- \( F^\times \) acts on \( \mathcal{S}'(F) \) by \( \langle a, \varphi \rangle = \langle a, \varphi_t \rangle \) iff \( \langle a, \varphi_t \rangle = \langle a, \varphi \rangle \).

\( \mathcal{S}'(F) = \{ a : \mathcal{S}(F) \to \mathbb{C} : \text{continuous linear form} \} \) is space of tempered distribution. For \( \chi : F^\times \to \mathbb{C}^\times \),
\[
\mathcal{S}'(F)^\chi = \{ a \in \mathcal{S}'(F) : \text{for all } t \in F^\times, a_t = \chi(t)a \}
\]
is the space of eigendistribution.

**Theorem 4.1.** For all \( \chi \in \Omega(F^\times) \), \( \dim \mathcal{S}'(F)^\chi = 1 \). (By Weil, in a Bourbaki talk on Tate’s thesis).

**Proof.** Recall that \( \varphi \in \mathcal{S}(F^\times) \) implies that \( M_{\varphi} \in \mathcal{S}(\Omega(F^\times)) \) (all polynomial functions on \( \Omega(F^\times) \).
\[
\mathcal{S}'(F)^\chi = \mathbb{C} \times \{ \text{evaluation of } M_{\varphi} \text{ at } \chi \},
\]
\[
\varphi \mapsto M_{\varphi}(\chi) = \int_{F^\times} \varphi(x)\chi(x)\mu^\chi(x).
\]
Here, \( \chi\mu^\chi \in \mathcal{S}'(F)^{\chi^{-1}} \).

Difference between \( \mathcal{S}'(F^\times) \) and \( \mathcal{S}'(F) \): For \( F \) non-archimedean, we have
\[
0 \to C_c^\infty(F^\times) \to C_c^\infty(F) \to \mathbb{C} \to 0
\]
and so we get an exact sequence
\[
0 \to (\mathbb{C})^\chi \to \mathcal{S}'(F)^\chi \to \mathcal{S}'(F^\times)^\chi.
\]
From this, we see that
\[
\dim \mathcal{S}'(F)^\chi \leq \begin{cases} 1 & \text{if } \chi \neq 1 \\ 2 & \text{if } \chi = 1 \end{cases}.
\]
For \( \chi \neq 1 \), \( \varphi \mapsto M_{\varphi}(\chi) \) defines a non-zero element of \( \mathcal{S}'(F)^\chi \). When \( \chi = 1 \), \( \varphi \mapsto \text{res}_{t=1} M_{\varphi} \) is a generator of \( \mathcal{S}'(F)^\chi \).
\( \mathcal{S}(\Omega(F^\times)) \) is invertible \( \mathcal{O} \)-module on \( \Omega(F^\times) \) and so \( \mathcal{S}(\Omega(F^\times))_\chi \) has dimension 1. \( \square \)

\[
\dim \mathcal{S}'(F)^\chi = 1, \ Re(\chi) > 0 (|t| < 1) \text{ then}
\]
\[
\langle \chi\mu^\chi, f \rangle = \int f(x)\chi(x)\mu^\chi(x)
\]
converges and \( \chi\mu^\chi \in \mathcal{S}'(F)^\chi \). Its Fourier transform \( f \mapsto \langle \chi\mu^\chi, \hat{f} \rangle \) is also a non-zero eigendistribution. \( \hat{\chi\mu^\chi} \in \mathcal{S}'(F)^{\chi_{w_1}} \) (\( \omega_1 \) is \( |\cdot| \)).
4.1. Eigendistribution and $\gamma$-function.

**Proposition 4.2.** Let $\chi \in \Omega(F^\times)$ then we can extend by zero to $\chi : F \to \mathbb{C}$. This function is locally integrable if $\text{Re}(\chi) > -1$, and differ by a tempered distribution $\chi \in \mathcal{S}'(F)\chi^{-1}\omega^{-1}$ where $\omega^{-1}(x) = |x|^{-1}$. Its Fourier transform $\hat{\chi} \in \mathcal{S}'(F)\chi$.

**Proof.** $\text{Re}(\chi) > -1$ implies that $\chi$ is locally integrable. Check case by case.

For $F = \mathbb{R}$,

$$\int_0^1 t^\sigma \mu(t) = (s + 1)^{-1} \left( t^{\sigma+1} \right)|_0^1$$

and so $t \mapsto t^\sigma$ is locally integrable if $\sigma > -1$.

For $\chi : F^\times \to \mathbb{C}^\times$, $|\chi| : F^\times \to \mathbb{R}_0^+$ is trivial on $O_F^\times$. Therefore, $|\chi|(x) = |x|^s$ for $s \in \mathbb{C}$. Define $\text{Re}(\chi) = \text{Re}(s)$. If $\chi$ is locally integrable, it defines a distribution $\chi : C_c^\infty(F) \to \mathbb{C}$. $\chi$ is tempered, thus linear functions can be extended $\mathcal{S}(F) \to \mathbb{C}$.

Let $\varphi \in \mathcal{S}(F)$,

$$\int_{F} \varphi(x) \chi(x) \mu(x) = \int_{|x| \leq 1} + \int_{|x| > 1}.$$

The $\int_{|x| \leq 1}$ converges as $\chi$ is locally integrable. $\int_{|x| > 1}$ converges because of the rapid decay $|\varphi(x)| x^\sigma x^\epsilon$ bounded and so $\chi$ is a tempered distribution.

$$\langle \chi, \varphi \rangle = \int_{F} \varphi(x) \chi(x) \mu(x)$$

converges.

$$\langle \chi_t, \varphi \rangle = \langle \chi, \varphi_{t^{-1}} \rangle = \int_{F} \varphi(tx) \chi(x) \mu(x) = \chi(t^{-1}) |t|^{-1} \langle \chi, \varphi \rangle.$$

This means that $\chi \in \mathcal{S}'(F)\chi^{-1}\omega^{-1}$. For $\chi \in \mathcal{S}'(F)$, Fourier transform

$$\langle \hat{\chi}, \varphi \rangle = \langle \chi, \hat{\varphi} \rangle.$$

For $\text{Re}(x) > -1$, $\hat{\chi}$ is a tempered distribution and $\hat{\chi} \in \mathcal{S}'(F)\chi$.

**Lemma 4.3.** For $t \in F^\times$, $\varphi \in \mathcal{S}(F)$, $(\hat{\varphi})_t = |t| \hat{\varphi}_{t^{-1}}$. If $a \in \mathcal{S}'(F)$, $(\hat{a})_t = |t|^{-1} \hat{a}_{t^{-1}}$.

$$\langle \hat{\chi}_t, \varphi \rangle = |t|^{-1} \langle \chi_{t^{-1}}, \varphi_{t^{-1}} \rangle = |t|^{-1} \chi_{t^{-1}}(t^{-1}) |t^{-1}|^{-1} \hat{\chi} = \chi(t) \hat{\chi}.$$

$\square$

**Proposition 4.4.** (Weil) For every character $\chi_1 : F^\times \to \mathbb{C}^\times$, $\dim \mathcal{S}'(F)\chi_1 = 1$.

**Proof.** For $\chi_1$ a fixed character, let $\chi \in \Omega(F)$ be the variable.

For $a \in \mathcal{S}'(F)\chi_1$, $a_1(\varphi) = \chi_1(t) a(\varphi)$ for all $\varphi \in \mathcal{S}(F)$. This means that $a (\varphi_{t^{-1}} - \chi_1(t) \varphi) = 0$. $a$ annihilates all functions of this form. Apply Mellin transform,

$$M (\varphi_{t^{-1}} - \chi_1(t) \varphi) (\chi) = \chi(t) M_{t} (\varphi) - \chi_1(t) M_{t} (\varphi)$$

Recall that $M : \mathcal{S}(F) \cong \mathcal{L} (\Omega(F))$. We can apply $a$ on both of these spaces. $a$ annihilates the submodule generated by $\langle \chi(t) - \chi_1(t) \rangle g$.

Therefore,

$$\dim \mathcal{L} (\Omega(F^\times)) / [\langle \chi(t) - \chi_1(t) \rangle g] = 1.$$

If $\chi_1 \notin \{\text{poles}\}$ then $a$ is a multiple of $g \mapsto g(\chi_1)$. If $\chi_1 \in \{\text{simple poles}\}$, $a$ is a multiple of $g \mapsto \text{res}_{x=\chi_1} g$. $\square$
When $\text{Re}(\chi) > -1$, $\chi \in \mathcal{S}(F)^\times \omega^{-1}$ and $\hat{\chi} \in \mathcal{S}(F)^\times$. If $-1 < \text{Re}(\chi) < 0$, then $\chi', \hat{\chi} \in \mathcal{S}(F)^\times$ where $\chi' = \chi^{-1}\omega^{-1}$. Since $\text{Re}(\chi) < 0$, $\text{Re}(\chi') > -1$. This gives two non-zero elements in the same one dimensional space. Write

$$\hat{\chi} = \gamma(\chi')$$

where $\gamma$ is a function of $\chi$, well defined for $0 < \text{Re}(\chi) < 1$.

The integral representation of $\gamma$ is given by the following,

$$\gamma(\chi)\chi'(y) = \int_F \chi(x)e(xy)\mu(x).$$

Let $y = 1$, then

$$\gamma(\chi) = \int_F \chi(x)e(x)\mu(x)$$

and so

$$\gamma(\chi) = \int_F \chi(x)e(x)|x|^{-1}\mu(x).$$

In particular, for $\chi(x) = |x|^s$,

$$\gamma(s) = \int_F e^{-2\pi i x|s|^{-1}}dx$$

which is very similar to

$$\Gamma(s) = \int_F e^{-\pi x|s|^{-1}}dx.$$ 

Anyways,

$$\gamma(\chi) = \int_{|x| \leq 1} + \int_{|x| > 1}$$

which both converge for $0 < \text{Re}(\chi) < 1$ and both have meromorphic continuation. Therefore, $\gamma$ has a meromorphic continuation.

To evaluate $\gamma$, evaluate $\hat{\chi} = \gamma(\chi)\chi'$ with a test function $\varphi \in \mathcal{S}(F)$. We have

$$M_{\varphi}(\chi) = \gamma(\chi)M_{\hat{\varphi}}(\chi^{-1}\omega).$$

Calculate $\gamma$ by choosing a nice test function.

Let $F = \mathbb{R}$, and let $\varphi$ be the basic function $b_\infty(x) = e^{-\pi x^2}$. Then $\hat{b}_\infty = e^{-\pi x^2}$.

$$Mb_\infty : \Omega(\mathbb{R}^\times) = \Omega_+ (\mathbb{R}^\times) \Pi \Omega_- (\mathbb{R}^\times) \to \mathbb{C}^\times.$$ 

The first component is generated by $|x|^s$ and the latter $\text{sgn}(x)|x|^s$.

$$Mb_\infty(\omega^s) = \pi^{-s/2}\Gamma(s/2)$$

and $0$ on $\Omega_-(\mathbb{R}^\times)$.

On $\Omega^+$,

$$\gamma(\omega^s) = \frac{Mb_\infty(\omega^s)}{Mb_\infty(\omega^{1-s})} = \frac{\pi^{-s/2}\Gamma(\frac{s}{2})}{\pi^{-(1-s)/2}\Gamma(\frac{1-s}{2})} = 2^{1-s}\pi^{s/2}\cos(\frac{\pi s}{2})\Gamma(s).$$

For $F = \mathbb{Q}_p$, calculate $\gamma$ on $\Omega^0(F^\times) = \mathbb{C}^\times$ the unramified characters. Recall, for $t \in \mathbb{C}^\times$, $\chi_t(\omega_p) = t$ and is trivial on $\mathcal{O}_F^\times$. Let $b_p = 1_{2p}$. Recall

$$Mb_p = \begin{cases} (1-t)^{-1} & \text{on } \Omega^0 \\ 0 & \text{else} \end{cases}$$

$$\gamma(\chi_t) = \frac{1-p^{-1}t^{-1}}{1-t}$$

and

$$\gamma(\omega^s) = \frac{1-p^{s-1}}{1+p^{-s}}.$$
5. Global Picture

Let $K$ be a global field, $A = A_K$. Let $\mathbb{A}^1 = \ker(\lvert \cdot \rvert : \mathbb{A}^\times \to \mathbb{R}_+^\times)$, and $K^\times \setminus \mathbb{A}^1$ is compact. Choose an Archimedean place of $K$ to get a splitting $\mathbb{A}^\times / K^\times = (\mathbb{A}^1 / K^\times) \times \mathbb{R}_+^\times$. Recall, we can normalize our measure so that $\text{vol}(\mathcal{O}_v^\times, \mu_v^\times) = 1$ when $v$ is non-Archimedean and $\mu_\infty^\times = \frac{2\pi}{\Pi}$ for Archimedean (self-dual).

$$\text{vol}(\mathbb{A}^1 / K^\times, \mu^\times) = \text{class number, regulator stuff.}$$

$\Omega(\mathbb{A}^\times / K^\times)$ is a countable union of $\Omega(\mathbb{R}_+^\times) = \mathbb{C}$.

For $\varphi \in \mathcal{S}(\mathbb{A})$ maps to $M_\varphi(\chi) = \int_{\mathbb{A}^\times} \varphi(x)\chi(x)\mu^\times(x)$. We will show that this converges for $\text{Re}(\chi) > 1$, have meromorphic continuous, and satisfy a similar functional equation.

We can get an invariant measure $\mu^1$ on $\mathbb{A}^1 / K^\times$ so that $\mu^\times = \mu^1 \times \mu_{\mathbb{R}_+^\times}$. For $\varphi$ functino on $\mathbb{A}^\times$,

$$\int_{\mathbb{A}^\times} \varphi(x)\mu^\times(x) = \int_{\mathbb{A}^\times / K^\times} \varphi_+(x)\mu^\times(x)$$

where $\varphi_+(x) = \sum_{\alpha \in K^\times} \varphi(\alpha x)$.

5.1. Zeta Integral. $\varphi \in \mathcal{S}(\mathbb{A})$ is a linear combination of $\otimes_v \varphi_v$ where $\varphi \in \mathcal{S}(K_v)$ and $\varphi_v = 1_{\mathcal{O}_v}$ for almost all non-archimedean $v$.

For $\chi \in \Omega_K = \Omega(\mathbb{A}^\times / K^\times) = \Omega(\mathbb{A}^1 / K^\times) \times \mathbb{C}$ (the first is a discrete group).

$$M_\varphi(\chi) = \int_{\mathbb{A}^\times} \varphi(x)\chi(x)\mu^\times(x).$$

**Proposition 5.1.** $M_\varphi(\chi)$ converges absolutely if $\text{Re}(\chi) > 1$ and define a holomorphic function on that region.

**Proof.** For $|\chi(x)| = |x|^\sigma$ for some $\sigma = \text{Re}(\chi)$,

$$\int_{\mathbb{A}^\times} |\varphi(x)| |x|^{\sigma} \mu^\times$$

is convergent for $\sigma > 1$.

Assume that $\varphi(x) = \prod_v \varphi_v(x_v)$ where $\varphi_v = 1_{\mathcal{O}_v}$ for all $v \in \vert K \vert - S$ for some finite set $S$. Suppose $v \notin S$, then

$$\int_{K_v^\times} 1_{\mathcal{O}_v}(x) |x|^{\sigma} \mu^\times(x) = (1 - q_v^{-\sigma})^{-1}$$

converges for $\sigma > 0$. If $\sigma > 1$, then $\prod_{v \notin S} (1 - q_v^{-\sigma})^{-1} < \infty$. This implies absolute convergence, holomorphic etc. \(\square\)

For all $\varphi \in \mathcal{S}(\mathbb{A})$, $M_\varphi$ holomorphic function $\Omega_K^{[\vert x \vert^\sigma > 1]}$ then $M_\varphi$ has meromorphic continuation and function equation.

**Proposition 5.2.** For every $x \in \mathbb{A}^\times$, $\bar{x}$ the image in $\mathbb{A}^\times / K^\times$, for all $\varphi \in \mathcal{S}(\mathbb{A})$, the series

$$\varphi_+(\bar{x}) = \sum_{\alpha \in K^\times} \varphi(\alpha x)$$

is absolutely convergent. Moreover, $\varphi_+(\bar{x})$ has rapid decay as $|\bar{x}| \to \infty$. That is, for $\sigma > 0$, $|\varphi_+(\bar{x})| |\bar{x}|^{\sigma}$ is bounded.

Assume $K = \mathbb{Q}$. $\mathbb{A}^\times = \mathbb{Q}^\times \times \mathbb{Z}_\mathbb{R}^\times \times \mathbb{R}_+^\times$. Can assume that $x = (x_{\mathbb{Q}}, x_\infty)$ where $x_{\mathbb{Q}} \in \mathbb{Z}_\mathbb{R}^\times$ and $x_\infty \in \mathbb{R}_+^\times$.

**Proof.** Suppose $\alpha x \in \text{supp}(\varphi)$, $x_p \in \mathbb{Z}_p$ and so $\alpha \in \mathbb{Z}_p$ (because support of $\varphi_v$ is $\mathbb{Z}_p$) and so $\alpha \in \mathbb{Z}$. This means

$$\varphi_+(x) = \sum_{\alpha \in \mathbb{Z} - \{0\}} \varphi_{\infty}(\alpha x).$$

$\varphi_\infty$ has rapid decay and so the series $\sum_{n \in \mathbb{Z} - \{0\}} \varphi(\alpha x)$ is convergent and has rapid decay as $x \to \infty$. \(\square\)
The analytic problem for \( \varphi \in \mathcal{S}(\mathbb{R}) \). We want to understand the Mellin transform of \( \varphi_+ \). Just take the positive \( \alpha \) for now, we see that

\[
\int_{\mathbb{R}_+^x} (\varphi(x) + \varphi(2x) + \ldots) x^s dx = (1 + 2^{-s} + 3^{-s} + \ldots) \int \varphi(x) x^s dx
\]

\[
= \zeta(s) \int \varphi(x) x^s dx.
\]

The problem is to understand the asymptotic behaviour of \( \varphi_+(x) \) as \( x \to 0 \). We can solve this by Poisson summation.

\[
\sum_{n \in \mathbb{Z}} \varphi(n) = \sum_{n \in \mathbb{Z}} \hat{\varphi}(n).
\]

For all \( x \neq 0 \),

\[
\sum_{n \in \mathbb{Z}} \varphi(nx) = |x|^{-1} \sum_{n \in \mathbb{Z}} \hat{\varphi}(nx^{-1})
\]

and so

\[
\sum_{n \in \mathbb{Z} - 0} \varphi(nx) = -\varphi(0) + |x|^{-1} \varphi(0) + \sum_{n \in \mathbb{Z} - (0)} \hat{\varphi}(nx^{-1})
\]

\[
= -\varphi(0) + |x|^{-1} \varphi(0) \text{ rapid decay}.
\]

The result is that as \( x \to 0 \),

\[\varphi_+(0) = -\varphi(0) + |x|^{-1} \hat{\varphi}(0) + O(x^\sigma)\]

(the first term contributes a pole at \( s = 1 \), the second at \( s = 1 \)).

Now for the general case,

\[
\int_{\mathbb{A}^x} \varphi(x) \chi(x) \hat{\mu}_x^\chi(x) = \int_{\mathbb{A}_x/K^x} \varphi_+(\tilde{x}) \chi(\tilde{x}) \hat{\mu}_x^\chi(x)
\]

makes sense when the integral converges absolutely for \( \sigma = \text{Re}(\chi) > 1 \).

\[
= \int_{|\tilde{x}| > 1} \varphi_+(\tilde{x}) \chi(\tilde{x}) \hat{\mu}_x^\chi(x) + \int_{|\tilde{x}| \leq 1} \varphi_+(\tilde{x}) \chi(\tilde{x}) \hat{\mu}_x^\chi(x).
\]

Since \( \varphi_+(\tilde{x}) \) has rapid decay as \( \tilde{x} \to \infty \), \( \int_{|x| > 1} \) is absolutely convergent for all \( \chi \). Meanwhile, \( \int_{|x| \leq 1} \) converges only when \( \sigma > 1 \). We will write

\[
\int_{|\tilde{x}| \leq 1} = (3) + (4) + (5)
\]

where 3 is absolutely convergent for all \( \chi \), 4 and 5 are simple functions that can easily be meromorphically continued.

For all \( \alpha \in \mathbb{A}^\times \), Poisson summation gives

\[
\varphi_+(x) = \sum_{\alpha \in K^\times} \varphi(\alpha x) \quad = |x|^{-1} \sum_{\alpha \in K^\times} \hat{\varphi}(\alpha x^{-1})
\]

\[
= -\varphi(0) + |x|^{-1} \hat{\varphi}(0) + |x|^{-1} \hat{\varphi}(x^{-1}) \text{ rapid decay as } x \to 0.
\]

Therefore,

\[
\int_{|\tilde{x}| \leq 1} = -\varphi(0) \int_{|x| \leq 1} \chi(x) \hat{\mu}_x^\chi(x) + \hat{\varphi}(0) \int_{|x| \leq 1} |x|^{-1} \chi(x) \hat{\mu}_x^\chi(x)
\]

\[
+ \int_{|x| > 1} \hat{\varphi}_+(x) |x|^{-1} \chi(x) dx.
\]
We can go further to compute this integral. Let \( \chi = \chi_1 \cdot |\cdot|^s \). Because \( \mathbb{A}^1/K^\times \) is compact:

\[
\int_{|x| \leq 1} \chi(x) \mu^\times(x) = \int_{\mathbb{A}^1/K^\times} \chi_1(x) \mu^1(x) \cdot \int_0^1 x^s \frac{dx}{x}
\]

\[
= \begin{cases} 
0 & \text{if } \chi_1 \neq 1 \\
\operatorname{vol}(\mathbb{A}^1/K^\times, \mu^1) \cdot \frac{1}{s} & \text{if } \chi_1 = 1
\end{cases}
\]

which converges for all \( s > 0 \). Similarly,

\[
(5) = \begin{cases} 
0 & \text{if } \chi_1 \neq 1 \\
\operatorname{vol}(\mathbb{A}^1/K^\times) \cdot \frac{1}{s-1} & \text{if } \chi_1 = 1
\end{cases}
\]

converges if \( \Re(s) > 1 \) and has meromorphic continuation.

Since this decomposition is symmetrical for \( \phi \leftrightarrow \hat{\phi} \) via \( \chi \leftrightarrow \omega \chi^{-1} \), we have

\[ M_\phi(\chi) = M_{\hat{\phi}}(\omega^1 \chi^{-1}) \]

which is global functional equation. Recall that our local functional equation is

\[ \gamma_v(\chi) M_{\phi_v}(\chi_v) = M_{\hat{\phi}_v}(\omega^\nu \chi^{-1}) \]

the global functional equation may be thought of as the fact that \( \prod \gamma_v(\chi) = 1 \). However, \( M_\phi \) converges for \( \Re(\chi) > 1 \) and \( M_{\hat{\phi}}(\chi) \) for \( \Re(\chi) < 0 \), so... can’t actually write this.

6. Langlands Conjectures

Two main problems in automorphic forms.

- Reciprocity conjectures
- Functoriality conjectures

They are all linked to \( L \)-functions.

6.1. Class Field Theory. Understand abelian extensions of number fields. Let \( K \) be a number field. It realizes \( G_K^{ab} \) with \( I_K = \mathbb{A}_K^\times/K^\times \) and local-global compatibilities.

Suppose \( v \) is a place of \( K \), then we have induced maps

\[ G_{K_v} \to G_K \text{ and } G_{K_v}^{ab} \to G_K^{ab} \]

which are well-defined up to conjugation.

6.1.1. Local Reciprocity. Let \( F = \mathbb{Q}_p \). If \( p \neq 2 \), there are three quadratic extensions, four if we count split extensions \( (\mathbb{Q}_p \times \mathbb{Q}_p) \).

Suppose \( K/F \) is a quadratic extension, then \( N_{K/F}(K^\times) \subseteq F^\times \) is a subgroup of index 2, so \( \chi_1 : F^\times \to F^\times/N_{K/F}(K^\times) = \{ \pm 1 \} \). We can check that \( F^\times/F^\times 2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

Write \( F^\times = \mathcal{O}_F^\times \times \pi_F^\mathbb{Z} \) where \( \pi_F \) is an uniformizer. Then

\[ F^\times/F^\times 2 = \mathcal{O}_F^\times/\mathcal{O}_F^{\times 2} \times \pi_F^{\mathbb{Z}}/\pi_F^{\mathbb{Z} 2} \cong \{ \pm 1 \} \times \{ \pm 1 \} \]

The first \( \pm 1 \) is generated by \( \tau \), a non-quadratic residue, and the second by \( \epsilon \), the class of \( \pi_F \).

The unramified extensions are those such that \( \chi : F^\times \to \mathbb{Z}/2\mathbb{Z} \) has \( \chi(\tau) = 1 \), and else ramified.
For an extension of number field $K/k$, we know that $K_v/k_v$ is unramified for almost all $v$. This means that the collection $\chi_v : k_v^\times \to \{\pm 1\}$ gives a quadratic character, which is an idele class character
\[
\chi : \mathbb{A}_k^\times \to k^\times \mathbb{A}_k^\times \to \{\pm 1\}.
\]
The idele class character $\chi$ has an associated $L$-function $L(\chi,1)$ which is a meromorphic function with complex parameter $s$. It is holomorphic if $\chi \neq 1$ and has a functional equation.

This can be extended to all characters $G_k \to \mathbb{C}^\times$.

6.1.2. **Galois representations.** Consider Artin representations $\sigma : G_k \to GL_n(\mathbb{C})$ irreducible. Then $\sigma$ must have finite image. Then $\sigma$ induces $\sigma_{k_v} : G_{k_v} \to GL_n(\mathbb{C})$.

\[
0 \to I_v \to G_{k_v} \to G_{k_v} \to 0
\]
where $G_{F_v} = \langle Fr_v \rangle \cong \mathbb{Z}$, $\sigma_{k_v}(Fr_v)$ is a well-defined conjugacy class of $GL_n(\mathbb{C})$.

\[
L^\sigma(\sigma, s) = \prod_{v \not\in S} \det(1 - \sigma_v(Fr_v)q_v^{-s})^{-1}
\]

**Conjecture 6.1.** (Artin) Can be extended to an holomorphic function of $s$ (non-trivial irreducible reps).

It has known meromorphic continuation. Langlands program in the early days, were to address this conjecture for many of theses cases (except icosahedral case).

We can define using local functions so that the complete $L$-function has functional equation $L(s, \sigma) = \epsilon L(1 - s, \sigma)$.

6.2. **Shimura-Taniyama-Weil Conjecture.** There are more representations $\sigma = G_k \to GL_n(\mathbb{Q}_p)$ coming from geometry, which can have infinite image. Just like Artin's $L$-function, we can write

\[
L_\sigma(A, s) = \prod_{v \not\in S} \det(1 - \sigma_v(Fr_v)q_v^{-s})^{-1}.
\]

This depends on a choice of identification $\overline{\mathbb{Q}}_p \cong \mathbb{C}$.

Taniyama explains that such representations coming from elliptic curves must be a Hecke $L$-function of a modular form of weight 2. These are automorphic representation of $GL_2(\mathbb{Q})$ whose archimedean component is given (a discrete series of $GL_2(\mathbb{R})$).

For $k = \mathbb{Q}$,

\[
GL_2(k)\backslash GL_2(\mathbb{A})/SO_2(\mathbb{R}) \cong \Gamma \backslash H.
\]

Discrete series means that $\varphi$ is a holomorphic function of $H$.

6.3. **Langlands conjecture.** Langlands: Reciprocity conjectures: Galois representations coming from $n$-dimensional representation ($\ell$-adic cohomology of algebraic variety) correspond to holomorphic reps of $GL_n$.

Functoriality: can be stated entirely within the automorphic theory

Suppose $G/k$ is a split reductive group. These are classified by Dynkin diagrams. The root data $\leftrightarrow$ dual root data $\to G^\vee/\mathbb{C}$.

$\pi = \otimes_{v \in |k|} \pi_v$ an automorphic representation of $G/k$, then $\pi_v$ is unramified for almost all $v$. $\pi_v^{G(C_v)} \neq 0 \leftrightarrow$ classified by semi-simple conjugacy class $\sigma_v \in G^\vee(\mathbb{C})$.

\[
\{v \notin S\} \leftrightarrow \{\sigma_v \in G^\vee(\mathbb{C})/\sim\}.
\]

Langlands introduced another data, $\rho : G^\vee \to GL_n(\mathbb{C})$ representation of $G^\vee$, and form the $L$-function

\[
L^\pi(s, \pi, \rho) = \prod_{v \notin S} \det(1 - \rho(\sigma_v)q_v^{-s})^{-1}.
\]
This has the same analytic properties as Riemann zeta-function (meromorphic continuations and functional equation).

6.4. Langland’s \( L \)-Group and automorphic \( L \)-functions. Let \( K \) be a perfect field, and \( T \) a torus. Let \( \Lambda = \text{hom}_k(\mathbb{G}_m, T) \cong \mathbb{Z}^r \) which has a continuous action of \( \Gamma = \text{Gal}(\overline{K}/K) \).

When \( r = 1 \), \( \text{Aut}(\mathbb{Z}) = \mu_2 \) and \( \Gamma \to \mu_2 \) is a quadratic extension of \( K \). When \( r > 1 \), we have a map \( \Gamma \to GL_r(\mathbb{Z}) \) (the latter being discrete).

There is an equivalence of categories

\[
\{ \text{torus over } k \} \iff \{ \text{free finite generated abelian group } \Lambda \text{ with finite action of } \Gamma \}.
\]

Let \( S_r \) be the symmetric subgroup of \( GL_r(\mathbb{Z}) \) in which \( \Gamma \) factors through (not necessarily surjects on). It is called the induced torus, and has an action of \( \Gamma \). It corresponds to \( K'/K \) finite etale extension of degree \( r \). \( K' = K[x]/P \) where \( \Gamma \) acts on the set of zeroes of \( P \).

From each \( \alpha \), can get a coroot \( \alpha^\vee : \mathbb{G}_m \to T \), where \( \langle \alpha, \alpha^\vee \rangle = 2 \). Let

\[
S_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha
\]

acting on \( \Lambda^\vee \), and

\[
S_\alpha(x) = x - \langle x, \alpha \rangle \alpha^\vee
\]

acting on \( \Lambda \).

The action generates a finite group \( W \) which stabilizer \( \Phi \) and \( \Phi^\vee \), called the Weil group.

Reductive groups over algebraically closed field/ism is equivalent to root datum.

Let \( \Phi^+ \) be so that

\[
\text{Lie} \ (B) = t \oplus \oplus_{\alpha \in \Phi^+} g_\alpha.
\]

There is a subset \( \Delta \subseteq \Phi^+ \) called simple roots (for \( GL_n \), the diagonal, just above the main diagonal). For all \( \alpha \in \Delta \), fix \( x_\alpha \in g_\alpha \) non-zero. The

\[
(T, B, \{x_\alpha\}_{\alpha \in \Delta})
\]

is pinning. The Normalizer of this is \( Z_G \) (center of \( G \)), so no more inner automorphisms.

**Proposition 6.2.** For \( G/K \) reductive, \( \text{Aut}(G) \) automorphism group of \( G \)

\[
0 \to \text{Inn}(G) = G/Z_G \to \text{Aut}(G) \to \text{Out}(G) \to 0
\]

has a splitting. The subgroup of \( \text{Aut}(G) \) stabilizing a pinning is isomorphic to \( \text{Out}(G) \).
The $K$-forms of $G$ is the set of reductive groups $G_K/K$, where $G_K \otimes_K \bar{K} \cong G$. It corresponds to $H^1(K, \text{Aut}(G))$. We have a long exact sequence
\[ H^1(K, \text{Inn}(G)) \to H^1(K, \text{Aut}(G)) \to H^1(K, \text{Out}(G)) \]
that has a lift. Note, $H^1(K, \text{Out}(G)) = \text{hom}(\Gamma_K, \text{Out}(G))$.

Reductive groups $G/K$ gives $\Psi = (A, \Phi, \Delta)$ with action of $\Gamma$. This gives $L_G$.

Let $\hat{G}/\mathbb{C}$ be such that its root system is $(\Lambda^\vee, \Phi^\vee)$. The action of $\Gamma$ on $\Psi$ gives rise to an action of $\Gamma$ on $\hat{G}$ (fixing pinning).

**Definition 6.3.** $L_G = \hat{G} \rtimes \Gamma$

Satake: $G$ split reductive group over $F$, non-archimedean local field. $K = G(O_F)$. Let $H(G) = C_c(K\backslash G/K)$ algebraic under convolution.

Gelfand proves that this is commutative.

Satake: $C_c(T(F)/T(O_F))^W = \mathbb{C}[\hat{T}]^W = \mathbb{C}[\hat{G}]^{\text{Ad}(\hat{G})}$ (the last equality is by Langlands).

Unramified representations of $G(F)$ correspond to irreducible representations with fixed values under $K$ correspond to simple $H$-modules. It is classified by
\[ \text{Spec} \left( \mathbb{C}[\hat{G}]^{\text{Ad}(\hat{G})} \right) = \text{semi-simple conjugacy class of } \hat{G}. \]

Now for the non-split case. Let $G$ be an unramified group of $F$ (so $\Gamma$ action on $\Psi$ is unramified). Then $L_G = \hat{G} \rtimes (\text{Frob})$.

Simple $H$-modules are the same as semi-simple conjugacy classes in $\hat{G} \cdot \text{Frob}$. This is exactly the equivalent class of splitting
\[ L_G \to (\text{Frob}). \]

Suppose
\[ \rho : L_G \to GL(V_\rho). \]

Then $L(s, \pi, \rho) = \prod_v \det \left( 1 - \rho(\sigma_v) q_v^{-s} \right)^{-1}$ like last time. It satisfies the same properties as Riemann zeta functions.