RESEARCH SUMMARY

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My work in the last few years is concerned with the study of the algebraic structures which underlay harmonic analysis over finite fields and over the reals and complex numbers. In this regard, my field of research is a part of "algebraic analysis" and may be referred to as "algebraic harmonic analysis". These algebraic structures appear in two forms which strongly interact with one another. The first is representation theory, specifically, the Weil representation of the symplectic group. The second is algebraic geometry, specifically, the theory of ℓ-adic sheaves in the finite field setting and the theory of algebraic D-modules in the real and complex setting. My research involves a purely mathematical aspect which is the systematic development of this algebraic framework and an applied mathematical aspect which is application of the mathematical theory to basic problems that originate from other parts of science such as physics and engineering.

Below, I will describe my work in more detail, trying to separate between the pure and the applied content.

0.1. **Pure mathematics.**

0.1.1. *The Weil representation over finite fields.* The theory of discrete harmonic analysis is the study of the discrete Fourier transform operator DFT acting on the space of complex valued functions on $\mathbb{F}^n$ where $\mathbb{F} = \mathbb{F}_p$ is the finite field with p elements. A basic non-trivial fact is that the DFT is a part of a family of operators which can be described formally as a unitary representation of the symplectic group $Sp(2n, \mathbb{F})$, called the *Weil representation*. This observation is powerful and allows a better understanding of the fine properties of the DFT. In the paper [1], we use the Weil representation in order to define a canonical basis of eigenfunctions for the DFT. The construction of this special basis is a particular case of a general construction of orthonormal bases which are associated with maximal tori in the symplectic group. In the first part of the paper [2], we give a comprehensive study of the theory of tori and their associated orthonormal bases in the setup of the Weil representation. The collection of these bases forms a dictionary of special functions which we refer to as the *oscillator dictionary* and functions in this dictionary are referred to as oscillator functions.

The oscillator functions are interesting objects of study on their own right, roughly, they share many properties which are characteristic to random functions. A quantitative study of these properties requires the use of algebraic geometry and is given in the second part of [2]. The main tool that we use is an algebra-geometric counterpart (an ℓ-adic sheaf) of the Weil representation, which we refer to as *the geometric Weil representation.*

In [3], we construct the geometric Weil representation. Our construction elaborates on a previous construction due to Deligne [11]. There are two main advantages

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of the geometric Weil representation. First it allows to treat split and non-split situations on an equal footing. Second, it allows to recast problems of representation theory and harmonic analysis in the language of algebraic geometry, and then to approach them using cohomological techniques.

In [4], we take advantage of these two attributes of the geometric Weil representation in order to prove the Kurlberg-Rudnick conjecture from arithmetic quantum chaos. This conjecture concerns the rate of convergence of the semi-classical limit of a certain matrix coefficients associated with eigenfunctions of quantized chaotic Hamiltonians in the Berry Hannay model of quantum mechanics.

When p is equal two, namely, the field $\mathbb{F}$ is of characteristic 2, the structure of the Weil representation changes dramatically and, in particular, it is no longer a representation of the symplectic group $Sp\left(2n,\mathbb{F}\right)$. This fact implies that the theory of harmonic analysis in characteristic 2 is different than in odd characteristic. In [5], we construct the Weil representation in characteristic 2. Our construction improves upon Weil's classical construction from [7] in two aspects: First, our version is a representation of a bigger group which contains Weil's pseudo symplectic group as a strict subgroup. Second, our construction is based on the method of canonical vector spaces, what makes it more functorial and transparent.

The traditional construction of the Weil representation is highly implicit. The ideology, due to Kazhdan, behind the formalism of canonical vector spaces is that there exists a natural vector space on which the action can be transparently described. In [6], the formalism of canonical vector spaces is developed for p odd. The statement is that the Weil representation is governed by a more fundamental object which is a quantization functor from the category of symplectic vector spaces over $\mathbb{F}$ to the Category of complex vector spaces. This approach to the Weil representation is explicit (no complicated formulas), moreover, it reveals additional algebraic structures which exists in the setting of discrete harmonic analysis which are not so evident from the traditional perspective.

The Weil representation of $SL\left(2,\mathbb{F}\right)$ and, more generally, the Weil representation of $SL\left(2,\mathbb{Z}/n\mathbb{Z}\right)$ for any $n\in\mathbb{N}$, is a fundamental object in number theory. In [8], we use the Weil representation of $SL\left(2,\mathbb{Z}/n\mathbb{Z}\right)$ in order to give a new proof of two classical results in number theory (both due to Gauss) - the quadratic reciprocity law and the sign of Gauss sum. Our approach should be contrasted with Weil's work [7], where he recasts several known proofs of the law of quadratic reciprocity in terms of the Weil representation of a certain cover of the group $SL(2,\mathbb{A}_{\mathbb{Q}})$ where $\mathbb{A}_{\mathbb{Q}}$ denotes the adele ring of $\mathbb{Q}$. In this regard, we show that quadratic reciprocity already follows from the Weil representation over finite rings and, moreover, we establish a conceptual mechanism, different from that of Weil, which implies the law of quadratic reciprocity.

0.1.2. *The Weil representation over the reals.* In the real setting, the Fourier transform is a particular operator in the Weil representation of the metaplectic[1] group $Mp\left(2n,\mathbb{R}\right)$. In addition, it is intimately associated with the Hilbertian space of Schwartz functions on $\mathbb{R}^n$. In [9], we develop a D-module theoretic interpretation of the Weil representation; in addition, we characterize the Schwartz space as the space of global solutions of a certain (twisted) D-module, which we refer to as the *Weil D-module*. This approach allows to define various local versions of the

[1]The metaplectic group $Mp\left(2n,\mathbb{R}\right)$ is a double cover of the symplectic group $Sp\left(2n,\mathbb{R}\right)$.

Schwartz space which we refer to as *local Schwartz spaces*. Another advantage of the D-module framework is that it allows to naturally incorporate powerful methods from homological algebra in the setting of real harmonic analysis. The main goal of [9] is to suggest an answer to a question of Deligne (see [10]) concerning the possible existence of a canonical pairing between certain pairs of local Schwartz spaces. The construction of the pairing uses the *Green class* of the Weil D-module, where, the notion of a Green class of a D-module is a generalization of the classical notion of Green form of a linear differential operator (see [13]). Another application of the Green class of the Weil D-module is to define reconstruction formulas generalizing the Fourier inversion formula (see below).

The Weil D-module constitutes the algebraic structure underlying real harmonic analysis, much in the same way that the Cauchy Riemann equations underlay complex analysis. The twist in the definition of the Weil D-module is the algebraic origin of the metaplectic sign (see [18]). Also, the Weil D-module should be considered as the real counterpart of the geometric Weil representation from the finite field setting.

0.2. **Applied mathematics.**

0.2.1. *Applications to digital signal processing.* The field of digital signal processing (DSP for short) studies the space of digital signals (finite sequences) which is the Hilbert space $\mathcal{H}$ of complex valued functions on $\mathbb{F}^n$, for $n \in \mathbb{N}$. The basic observation is that this space should be considered as a representation space acted upon by the group $Sp\left(2n, \mathbb{F}\right)$ via the Weil representation

$$\rho : Sp\left(2n, \mathbb{F}\right) \rightarrow U\left(\mathcal{H}\right).$$

In this regard, the Weil representation establishes the bridge between DSP and representation theory. A basic problem in DSP is to construct interesting and useful dictionaries of signals which satisfy various properties which appear in the engineering wish list. There are few mathematical constructions of such dictionaries; representation theory is one of these few. In [2], we use the Weil representation in order to construct the oscillator dictionary (see more details above); we prove that the oscillator functions satisfy properties which makes them ideal for applications to the theory of discrete radar (see [16]) and communication theory (see [15]), particularly CDMA systems.

The oscillator dictionary has a particular form, its a disjoint union of orthonormal bases such that functions from different bases are almost orthogonal to one another. Such a system of orthonormal bases is called an *incoherent system*. Incohernt systems are rigid objects and seem to play a fundamental role in DSP; there are few mathematical constructions of such systems and in most constructions that we know the incoherence property is accounted by deep results from algebraic geometry. In [12], we study the statistical properties of incoherent systems, in particular, we prove that an incoherent system satisfy a statistical version of the restricted isometry property (RIP for short) which appears to be useful in applications related to sparsity.

0.2.2. *Applications to analog signal processing.* The field of analog signal processing (ASP for short) studies the space of analog signals which is the Hilbertian space $\mathcal{S}$ of Schwartz functions on $\mathbb{R}^n$, for $n \in \mathbb{N}$, which should be considered as a representation space of the metaplectic group $Mp\left(2n,\mathbb{R}\right)$ acting via the Weil representation

$$\rho : Mp\left(2n,\mathbb{R}\right) \rightarrow U\left(\mathcal{S}\right).$$

An analog signal φ can be effectively reconstructed from its Fourier transform $\widehat{\varphi}$ using the Fourier inversion formula. More generally, it can be effectively reconstructed from $\varphi^{g}=\rho\left(g\right)\varphi$, for any $g\in Mp\left(2n,\mathbb{R}\right)$ as $\varphi=\rho\left(g^{-1}\right)\varphi^{g}$. We are currently studying general reconstruction formulas which can be defined using the Green class of the Weil D-module [9] (see above). These formulas are cohomological in nature and constitute a far reaching generalization of the Fourier inversion formula and its variants. Very roughly, the result is that a signal φ can be **effectively** reconstructed from the collection of values $\left\{\varphi^{g}\left(x\right):\left(g,x\right)\in X\right\}$ where X is any submanifold of dimension n of $Mp\left(2n,\mathbb{R}\right)\times\mathbb{R}^{n}$, satisfying some mild properties.

References

[1] Gurevich S. and Hadani R., On the diagonalization of the discrete Fourier transform. To appear in Applied and Computational Harmonic Analysis (2008).

[2] Gurevich S., Hadani R. and Sochen N., The finite harmonic oscillator and its application to sequences, communication and radar. To appear in the IEEE Transaction on Information Theory (2008).

[3] Gurevich S. and Hadani R., The geometric Weil representation. Selecta Mathematica, New Series, 13 (3), 465-481 (2007).

[4] Gurevich S. and Hadani R., Proof of the Kurlberg-Rudnick rate conjecture. Submitted to the Annals of Mathematics in March 2005.

[5] Gurevich S. and Hadani R. The Weil representation in characteristic two. Submitted to Advances In Mathematics (2008).

[6] Gurevich S. and Hadani R., Quantization of symplectic vector spaces over finite fields. Submitted to Selecta Mathematica (2008).

[7] Weil A., Sur certains groupes d'operateurs unitaires. *Acta Math. 111* (1964), 143-211.

[8] Gurevich S., Hadani R. and Howe R., Quadratic reciprocity and sign of Gauss sum via the finite Weil representation. To appear IMRN (2008).

[9] Hadani R., The Weil representation over the complex numbers and localization of Schwartz spaces. In preparation.

[10] Deligne P., Groupe de Heisenberg et rèalité (French), *J. Amer. Math. Soc.* **4** (1991), no 1, 197–206.

[11] Deligne P., Metaplectique. *A letter to Kazhdan* (1982).

[12] Gurevich S. and Hadani R., Incoherent dictionaries and the statistical restricted isometry property. Submitted to Applied and Computational Harmonic Analysis (2008).

[13] Gonchaorov A., Differential equations and integral geometry, *Adv. Math.* **131** (1997), no. 2, 279–343.

[14] Beilinson A., Bernstein J., A proof of Jantzen conjecture. I.M. Gelfand seminar, 1–50, *Adv. Soviet Math.,***16** part 1

[15] Golomb S.W. and Gong G., Signal design for good correlation. For wireless communication, cryptography, and radar. *Cambridge University Press, Cambridge* (2005).

[16] Howard S. D., Calderbank A. R. and Moran W., The finite Heisenberg-Weyl groups in radar and communications. *EURASIP J. Appl. Signal Process.* (2006).

[17] Howe R., Dual pairs in physics: harmonic oscillators, photons, electrons, and singletons. *Applications of group theory in physics and mathematical physics* (Chicago 1982), 179-207, *Lectures in appl. Math.,* **21**, *Amer. Math. Soc., providence,* RI (1985).

[18] Lion G., Vergne M., The Weil representation, Maslov index and theta series. *Progress in Mathematics,* **6**. Birkhäuser, Boston, Mass (1980).

[19] Weinstein A., The symplectic "category". *Differential geometric methods in mathematical physics. Lecture Notes in Math., 905, Springer, Berlin-New York* (1982) 45–51.