# MIN-MAX FOR THE ALLEN-CAHN EQUATION AND OTHER TOPICS (PRINCETON, 2019)

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ABSTRACT. The following is a list of problems, substantial hints and brief discussions, related to the topics of my talk during the Summer School in Geometric Analysis at Princeton University in June of 2019. The focus is on elementary properties of solutions that are independent of the dimension, as well as min-max and other methods for constructing solutions.

## 1. NOTATION FOR HYPERSURFACES

**Definition 1.1.** Given $\Sigma \subset M$, we denote by $|\Sigma| := \mathcal{H}^{n-1}(\Sigma)$ the $(n-1)$-dimensional Hausdorff measure of the set $\Sigma$, i.e. $|\Sigma| = \text{Area}(\Sigma)$, for $\Sigma = \Sigma^{n-1}$ a smooth embedded hypersurface.

**Definition 1.2.** Let $\Sigma \subset M$ be an embedded smooth hypersurface. We say that $\Sigma$ is *separating* iff $M \setminus \Sigma$ is the union of two open regions $M_1$ and $M_2$ (not necessarily connected) and such that $\Sigma = \partial M_1 = \partial M_2$.

**Example 1.3.** The union of $k$ disjoint slices (copies of $\mathbb{T}^{n-1}$) of a torus $\mathbb{T}^n$ form a separating hypersurface iff $k$ is even. $\mathbb{RP}^2 \subset \mathbb{RP}^3$ is not separating. Indeed, since we are assuming $M$ is orientable, then $\Sigma$ cannot be separating whenever one of its connected components is non-orientable.

**Definition 1.4.** We say that an embedded minimal hypersurface $\Sigma_0$ (possibly unorientable) is *non-degenerate* iff $\frac{d^2}{dt^2}\text{Area}(\Sigma_t)|_{t=0} \neq 0$, for all smooth variations $\Sigma_0$, such that $\frac{d}{dt}\Sigma_t|_{t=0}$ is normal to $\Sigma_t$ and not identically zero. If in addition, $\frac{d^2}{dt^2}\text{Area}(\Sigma_t)|_{t=0} \geq 0$ for every variation, we say that $\Sigma$ is *stable* (or *strictly stable* if the inequality is strict).

**Remark.** Let $\Sigma$ be an orientable minimal hypersurface and $\partial$ a continuous choice of a normal vector. If $\frac{d}{dt}\Sigma_t|_{t=0} = f\partial$, for $f \in C^\infty(\Sigma)$, then $\frac{d^2}{dt^2}\text{Area}(\Sigma_t)|_{t=0} = -\int_M fJ(f)$, where $J = \Delta_\Sigma + |A_0|^2 + \text{Ric}_M(\partial, \partial)$ is the Jacobi operator of $\Sigma$ and $A_0$ is the second fundamental form of $\Sigma$ (see [4]). In this case, we denote by $\text{Ind}(\Sigma)$ the number of negative eigenvalues of $-J$ (counted with multiplicity).

## 2. THE ALLEN-CAHN EQUATION

Let $\Omega \subset M^n$ be a region with smooth boundary, of an $n$-dimensional Riemannian manifold). Given $\varepsilon > 0$, we refer to the semilinear elliptic equation

$$\varepsilon^2 \Delta u - W'(u) = 0, \tag{1}$$

where $u : \Omega \to \mathbb{R}$ is a $C^2(\Omega)$ function and $W(u) = (1-u^2)^2/4$, as the Allen-Cahn equation.

Notice that the constants $\pm 1$ and $0$ satisfy (1). We refer to these as *trivial* solutions of (1). A simple computation shows that (1) is the Euler-Lagrange equation of the energy

$$E_\varepsilon(u) = \int_M \varepsilon \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} \tag{2}$$

as a functional on the Sobolev space $W^{1,2}(M)$, i.e. $u$ solves (1) iff it is a critical point of $E_\varepsilon$.

**Definition 2.1.** If $u$ solves (1), we denote by $\mathrm{Ind}(u)$ the number of negative eigenvalues of the linearized operator $-L_u$ where $L_u = \varepsilon^2\Delta - W''(u)$. Notice that $E''_\varepsilon(u)(\phi,\phi) = -\int_M \phi L_u \phi$.

**Exercise 1** ($\varepsilon$-scaling)**.** *If $u$ satisfies* (1) *on* $(M, g)$*, then $u$ satisfies* (1) *with* $\varepsilon = 1$ *on the rescaled metric* $(M, \varepsilon^{-2}g)$*.*

**Exercise 2** ($\pm 1$ are the solution with lowest energy)**.** *Show that* $\pm 1$ *are the only global minimizers and* $E_\varepsilon(\pm 1) = 0$*.*

**Exercise 3** ($0$ is the solution with highest energy)**.** *Show that if $u$ is any non zero solution of* (1) *(not necessarily trivial) then* $E_\varepsilon(u) < E_\varepsilon(0)$*.*

*Hint: integrate by parts in* (2)*, substitute* $\varepsilon^2\Delta u = u^3 - u$ *and simplify.*

**Exercise 4** (The 1-D canonical solution)**.** *On* $\mathbb{R}$*, there is only one entire non-trivial solution to* (1) *with finite energy given by* $\psi(t/\varepsilon) = \tanh\left(\frac{t/\varepsilon}{\sqrt{2}}\right)$ *(modulo translations and reflections). The energy constant* $E_\varepsilon(\psi)$ *does not depend on* $\varepsilon$ *and we denote it by* $\sigma = E_\varepsilon(\psi)$*.*

## 3. Connection to minimal hypersurfaces

Informally, we expect $\{u = 0\}$ to converge to a minimal hypersurface $\Sigma^{n-1} \subset M^n$ (which is perhaps singular). Moreover, $\lim_{\varepsilon\to 0} E_\varepsilon(u) = \sigma|\Sigma|$. There are elementary heuristic arguments supporting such expectations (examples can be found on [13] and `math.stanford.edu/~ryzhik/BANFF/delpino.pdf`). The following are precise versions of this fact.

**Theorem 3.1** (Pacard-Ritoré [12] (see also [11, 2]))**.** *Given* $\Sigma \subset M$ *a smooth, separating, non-degenerate minimal hypersurface, there exists* $\varepsilon_0 > 0$*, such that for every* $\varepsilon \in (0, \varepsilon_0)$ *there is a solution $u$ to* (1) *such that* $\{u = 0\} \to \Sigma$ *as a smooth graph. Moreover,* $\lim_{\varepsilon\to 0} E_\varepsilon(u) = \sigma|\Sigma|$ *and* $\mathrm{Ind}(\Sigma) = \mathrm{Ind}(u)$*.*

**Theorem 3.2** (Hutchinson-Tonegawa-Wickramasekera-Guaraco [8] (see also [9, 14]))**.** *Assume there is a sequence of solutions $u = u_k$ to* (1)*, with* $\varepsilon = \varepsilon_k \to 0$*, such that*

$$\sup_\varepsilon \left(\|u\|_{L^\infty(\Omega)} + E_\varepsilon(u) + \mathrm{Ind}(u)\right) < +\infty, \tag{3}$$

*then* $\{u = 0\}$ *converges (with respect to the Hausdorff distance) to a minimal hypersurface* $\Sigma$*, which is embedded outside of a set of dimension at most* $n - 8$*.*

*Moreover, if* $\Sigma_1, \dots, \Sigma_k$ *are the connected components of* $\Sigma$*, then* $\lim_{\varepsilon\to 0} E_\varepsilon(u) = m_1|\Sigma_1| + \cdots + m_k|\Sigma_k|$*, for* $m_i \in \mathbb{N}$*.*

**Remark.** It is possible to define an index for $\Sigma$ with respect to vector fields. This works even when it is not orientable. In this sense, P. Gaspar [5] showed that under hypothesis (3), one has $\mathrm{Ind}(\Sigma) \leq \lim_{\varepsilon\to 0} \mathrm{Ind}(u)$.

**Theorem 3.3** (Chodosh-Mantoulidis [3])**.** *Let* $n = 3$*. Assume there is a sequence of solutions* $u = u_k$ *to* (1)*, with* $\varepsilon = \varepsilon_k \to 0$*, such that*

$$\sup_\varepsilon \left(\|u\|_{L^\infty(\Omega)} + E_\varepsilon(u) + \mathrm{Ind}(u)\right) < +\infty,$$

*then, outside of a finite set,* $\{u = 0\}$ *converges to a minimal hypersurface* $\Sigma$ *as a multigraph. As above,* $\lim_{\varepsilon\to 0} E_\varepsilon(u) = m_1|\Sigma_1| + \cdots + m_k|\Sigma_k|$*. If* $m_i \neq 1$*, then* $\Sigma_i$ *admits a positive Jacobi vector field. In particular, if* $\Sigma$ *is non-degenerate or* $\mathrm{Ric}_M$ *is positive then* $m_i = 1$*, for all* $i = 1, \dots, k$*.*

**Remark.** Let $\Sigma$ be a separating hypersurface. Denote by $\mathrm{Nul}(\Sigma)$ and $\mathrm{Nul}(u)$, the nullity of $J$ and $E''_\varepsilon$, respectively. Chodosh-Mantoulidis [3] showed that if $m_i = 1$ for all $i = 1, \dots, k$ then $\lim_{\varepsilon\to 0} \mathrm{Ind}(u) + \mathrm{Nul}(u) \leq \mathrm{Ind}(\Sigma) + \mathrm{Nul}(\Sigma)$.

## 4. Other fundamental properties of solutions

Let $M$ be a closed manifold and $u$ a solution to (1) on $M$.

**Exercise 5.** *Show that* $|u| \leq 1$*. If the equality holds at one point then* $u = \pm 1$*.*
*Hint: use the maximum principle.*

**Exercise 6.** *If* $u$ *is a non-trivial solution then* $\{u = 0\} \neq \emptyset$*.*
*Hint: Assume* $\{u = 0\} = \emptyset$ *and try the linear deformation connecting* $u$ *with* $\mathrm{sgn}(u)$*, i.e* $(1-t)u + t\,\mathrm{sgn}(u)$*.*

**Exercise 7** (*)**.** *Show that if* $u(p) = 0$ *then* $E_\varepsilon(u)|_{B_\varepsilon(p)} > c_0$ *for some universal constant* $c_0$*.*
*Hint: Work on the rescaled metric* $(M, \varepsilon^{-2}g)$*. There,* $u$ *satisfies* $\Delta u - W'(u) = 0$*. Since* $|u| \leq 1$ *you can use Schauder estimates to obtain* $C^{1,\alpha}$*-bounds on* $u$*. This implies that the* $u$ *must be close to zero on* $B_1(p)$*. Scaling back, this gives a lower bound on the potential term of the energy* $\int_{B_\varepsilon(p)} W(u)/\varepsilon$*.*

**Exercise 8** (A positive solution with Dirichlet boundary data exists if $\varepsilon$ is small or if the region is large enough)**.** *Let* $\phi$ *be the first eigenfunction of the Laplacian on a region* $\Omega$*. Show that* $E_\varepsilon(\phi) < E_\varepsilon(0)$ *iff* $\varepsilon^2\lambda_1 < 1 - \frac{1}{2}\frac{\int_M \phi^4}{\int_M \phi^2}$*. Conclude that if* $\varepsilon^2\lambda_1 < 1/2$ *then there is a positive solution to* (1) *with Dirichlet boundary data on* $\Omega$*.*
*Hint: By standard compactness methods (Rellich-Kondrachov) you can minimize* $E_\varepsilon$ *on* $W^{1,2}_0(\Omega)$*. The bound* $E_\varepsilon(\phi) < E_\varepsilon(0)$ *guarantees that the minimizer* $u$ *is not zero. Since* $|\nabla|u|| = |\nabla u|$*, it follows that* $|u|$ *is also a minimizer. Use the maximum principle to conclude that* $u$ *is non-zero in* $\Omega$*.*

**Exercise 9** (*Uniqueness of positive solutions with Dirichlet boundary data)**.** *Show that the solution from the previous exercise is unique.*

*Hint: This proof from* [1], *is general and it works for a wide range of semilinear equations. Let $u_1$ and $u_2$ positive solutions with Dirichlet boundary data. The formula $-W'(u_i) < u_i$, implies $0 < \Delta u_i + u_i$. Apply the strong maximum principle to conclude $\partial_\nu u_i < 0$ and that the functions $u_1/u_2$ and $u_2/u_1$ are in $L^\infty$. Integrating by parts obtain the formula*

$$\int_\Omega \left( -\frac{\Delta u_1}{u_1} + \frac{\Delta u_2}{u_2} \right)(u_2^2 - u_1^2) = \int_M \left| \nabla u_1 + \frac{u_1}{u_2} \nabla u_2 \right|^2 + \left| \nabla u_2 + \frac{u_2}{u_1} \nabla u_1 \right|^2 \geq 0.$$

*Then uniqueness, i.e. $u_1 = u_2$, follows from $\int_\Omega \left( -\frac{W'(u_1)}{u_1} + \frac{W'(u_2)}{u_2} \right)(u_2^2 - u_1^2) \geq 0$.*

**Remark.** On $\Omega \setminus \{u = 0\}$, we can rewrite the Allen-Cahn equation as a linear equation with good decay estimates

$$\begin{aligned} 0 &= \varepsilon^2 \Delta u - W'(u) \\ &= \varepsilon^2 \Delta (u - \operatorname{sgn}(u)) - W'(u) \\ &= \varepsilon^2 \Delta (u - \operatorname{sgn}(u)) - u^3 + u \\ &= \varepsilon^2 \Delta (u - \operatorname{sgn}(u)) - u(u^2 - 1) \\ &= \varepsilon^2 \Delta (u - \operatorname{sgn}(u)) - u(u^2 - \operatorname{sgn}(u)^2) \\ &= \varepsilon^2 \Delta (u - \operatorname{sgn}(u)) - u(u + \operatorname{sgn}(u))(u - \operatorname{sgn}(u)) \\ &= \varepsilon^2 \Delta (u - \operatorname{sgn}(u)) - |u|(|u|+1)(u - \operatorname{sgn}(u)) \\ &= \varepsilon^2 \Delta v - cv, \end{aligned}$$

where $v = u - \operatorname{sgn}(u)$ and $c = |u|(|u|+1) \geq 0$. The formula $0 = \varepsilon^2 \Delta v - cv$ is a linear equation with $c > 0$. Exponential decay for solutions of this equation are standard. Indeed:

**Exercise 10** (*)**.** *Show that the function $v = u - \operatorname{sgn}(u)$ decays exponentially fast in terms of its distance to the nodal set $\{u = 0\}$. More precisely, $|v(p)| \leq Ce^{-\sigma t/\varepsilon}$, where $t = \operatorname{dist}_M(p, \{u = 0\})$.*

*Hint: For the canonical 1-D solution, this follows immediately from its formula. In general, you can work on a region where the function $t = \operatorname{dist}_M(\cdot, \{u = 0\})$ is smooth. Here, you can construct supersolutions to the linear operator $\varepsilon^2 \Delta - c$, where $c$ is the function above. Try functions of the form $ae^{-\sigma t/\varepsilon} + b\cosh(t/\varepsilon)$ in order to apply the maximum principle on a bounded set. Then, make $b \to 0$.*

**Definition 4.1.** We say that a function $v = v_\varepsilon$ is of order $o(\varepsilon^{\mathbb{N}})$ on a region $\Omega$ if all of its derivatives and integrals on $\Omega$ decay faster than any polynomial, i.e. $\|\nabla^k v\|_{L^p(\Omega)} = o(\varepsilon^m)$, for all $m \in \mathbb{N}$.

**Remark.** By Schauder, the exponential decay extends to all the derivatives of $u - \operatorname{sgn}(u)$, in particular, $|\nabla^k u| \leq Ce^{-\sigma t/\varepsilon}$, where $t = \operatorname{dist}_M(p, \{u = 0\})$. This formula implies precise estimates for $u - \operatorname{sgn}(u)$ and its derivatives on regions where $t/\varepsilon > L > 0$. Moreover, $u - \operatorname{sgn}(u)$ is $o(\varepsilon^{\mathbb{N}})$ in regions where $t/\varepsilon \to \infty$. Estimates on regions where $t/\varepsilon \leq L$, follow from blow-up arguments depending on particular geometrical assumptions. These ideas are a central aspect of the theory.

**Exercise 11.** *Let $B_R(0)$ be the $n$-dimensional ball of radius $R > 0$ in $\mathbb{R}^n$. Show that if $u$ is the positive solution to* (1) *on $B_R(0)$ then $u$ is rotationally symmetric and $1 - Ce^{-R/\varepsilon} \leq u(0) < 1$. Conclude that when $R \to \infty$ then $u \to +1$ on compacts.*

*Hint: for the last claim use Schauder estimates and compactness embeddings of Holder spaces.*

## 5. Construction of solutions by min-max

Since $E_\varepsilon$ has only two isolated global minima, it is natural to expect the existence of a solution of mountain-pass type with $\mathrm{Ind}(u) \leq 1$. Moreover, by the convergence theorems from Section 3, if we can control the energy of such solutions, both from above and below, we can construct a minimal hypersurface (perhaps with a singular set of dimension at most $n-8$). This was done originally in [8] for $n \geq 3$ and extended to $n = 2$ on [10]. In this section we discuss a generalization of this construction which is presented in detail in [6].

**Exercise 12.** *Show that there is a Morse function $f : M \to \mathbb{R}$ such that the level sets $\Sigma_t = \{f = t\}$ move continuously with respect to the Hausdorff distance.*

*Hint: the problem is that $f$ might have local maxima (or minima). Nonetheless, these are isolated points and you can transform them into global maxima (or minima) by modifying the function just on a small neighborhood around this points.*

Let $f$ be a Morse function as in the previous exercise. As a first step on the construction of solutions, we use the one-parameter family of hypersurfaces $\Sigma_t = \{f = t\}$ to give an example of a higher-dimensional, odd family of functions $h : S^p \to W^{1,2}(M)$ with energy bounded from above independently of $\varepsilon$.

For each $z \in \mathbb{C}$, we define an associated distance function on $M$ by

$$d_z : M \to \mathbb{R}_{\geq 0}, \quad d_z(x) = \mathrm{dist}_M(x, \Sigma_{\mathrm{Re}(z)}) + \mathrm{dist}_{\mathbb{C}}(z, f(M)),$$

where $f(M) = [\min_M f, \max_M f]$. For each $a = (a_0, \ldots, a_p) \in S^p$, consider the polynomial $P_a(z) = \sum_{i=0}^p a_i z^i$ and let $C(a)$ be the set of its roots in the complex plane. We then define the functions

$$d_a(x) = \begin{cases} \min\{d_z(x) : z \in C(a)\} & \text{if } C(a) \neq \emptyset \\ +\infty & \text{if } C(a) = \emptyset \end{cases}.$$

Finally, define

$$\rho_a(x) = \mathrm{sgn}_a(x) d_a(x),$$

where $\mathrm{sgn}_a(x) = \mathrm{sgn}(P_a \circ f(x))$, whenever $d_a(x) > 0$.

**Exercise 13** (**). *Let $\psi$ be the canonical one-dimensional solution. Show that $a \in S^p \to h_a = \psi \circ \rho_a \in W^{1,2}(M)$ is continuous. Moreover, $h_{-a} = -h_a$ and $\limsup_{\varepsilon \to 0} \sup_\varepsilon E_\varepsilon(h_a) = \sigma p \sup_{t \in \mathbb{R}} |\Sigma_t|$.*

*Hint: once we have constructed the right distance functions $d_a$, the rest of the computations are long, but elementary. You can find them on* [6].

The topology of $W^{1,2}(M)$ is trivial, but to hope to find critical points by means of min-max methods we need first to have some interesting topology. In this particular case, this follows from equivariant min-max methods that exploit the $\mathbb{Z}_2$-symmetries of the functional $E_\varepsilon$ on $W^{1,2}(M)$.

Define $\mathcal{H} = W^{1,2}(M) \setminus \{0\}$ and denote by $\mathcal{S}$ the unit sphere of $\subset \mathcal{H}$. Then $\mathcal{H} \simeq \mathcal{S} \times (0, +\infty)$ is a (non-complete) smooth Hilbert manifold. We can think of the $\mathcal{S}$ as a infinite dimensional sphere. More precisely, given the natural inclusions of finite dimensional spheres $S^1 \subset S^2 \subset S^3 \subset \cdots$ the infinite dimensional sphere is defined as the set $S^\infty = \cup_{k\in\mathbb{N}} S^k$ with the largest topology making the inclusions continuous. Given an infinite set of linearly independent vectors $v_1, v_2, v_3, \ldots$ on $W^{1,2}$ (e.g. eigenvalues of the Laplacian) we can take the unit sphere on each finite dimensional span $E^k = span(v_1, \ldots, v_k)$. This gives us the inclusions $S^1 \subset S^2 \subset \cdots \subset S^\infty \subset \mathcal{H} \simeq \mathcal{S} \times (0, +\infty)$.

Now, notice that the $\mathbb{Z}_2$-action on $\mathcal{H}$ given by the antipodal map $u \to -u$ is free. Therefore, $\mathcal{H}/\mathbb{Z}_2$ is a smooth (but punctured) Hilbert manifold and we have the inclusions

$$\mathbb{RP}^1 \subset \mathbb{RP}^2 \subset \cdots \subset \mathbb{RP}^\infty \subset \mathcal{H}/\mathbb{Z}_2 \simeq \mathcal{S}/\mathbb{Z}_2 \times (0, +\infty).$$

It is a well-known fact that the cohomology of $\mathbb{RP}^\infty$ (with coefficients $\mathbb{Z}_2$) is a polynomial algebra $\mathbb{Z}_2[\gamma]$ generated by a single non-trivial element $\gamma$ of the first cohomology group. It is the case that the cohomology group of $\mathcal{H}/\mathbb{Z}_2 \simeq \mathcal{S}/\mathbb{Z}_2 \times (0, +\infty)$ has the same form. This is a more subtle assertion (see [6] for the precise topological statements). However, the inclusions above should at least convince you that the cohomology of $\mathcal{H}/\mathbb{Z}_2$ is at least as complicated as $\mathbb{Z}_2[\gamma]$.)

Summarizing, all these arguments tell us that it should be possible to apply min-max methods to find critical points of $E_\varepsilon$ in $\mathcal{H}/\mathbb{Z}_2$ (and therefore in $\mathcal{H}$) as long as we can guarantee that our min-max families stay on a bounded set and also away from the singular point which is the origin.

Given $p \in \mathbb{N}$, we define the family

$$\mathcal{F}_p = \{A \subset W^{1,2}(M) : A \text{ compact}, A = -A \text{ and } \gamma^p(A/\mathbb{Z}_2) \neq 0\}.$$

Finally, we define the $p$-widths

$$c_\varepsilon(p) = \inf_{A \in \mathcal{F}_p} \sup_A E_\varepsilon.$$

From the inclusions $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \mathcal{F}_2 \supset \cdots$ it follows that $c_\varepsilon(1) \leq c_\varepsilon(2) \leq \cdots$ is a monotone increasing sequence.

In [6], we use that $E_\varepsilon$ has a Palais-Smale property on a suitable subset of $\mathcal{H}$, from classical min-max theory to conclude the following result:

**Theorem 5.1.** *Fix $p \in \mathbb{N}$. For $\varepsilon$ small enough, there exists a non-trivial solution $u$ to* (1) *with $E_\varepsilon(u) = c_\varepsilon(p) < E_\varepsilon(0)$ and $\mathrm{Ind}(u) \leq p \leq \mathrm{Ind}(u) + \mathrm{Nul}(u)$. Moreover,*

$$0 < \liminf_{\varepsilon\to 0} c_\varepsilon(p) \leq \limsup_{\varepsilon\to 0} c_\varepsilon(p) < +\infty.$$

*Proof.* Fix $p \in \mathbb{N}$. Let $h$ be as in the previous exercise. The existence of $\varepsilon$, follows since $h(S^p) \subset \mathcal{F}_p$, combined with the upper bound on $E_\varepsilon(h_a)$ (which is independent of $\varepsilon$) and the fact that $E_\varepsilon(0) = \varepsilon^{-1}\mathrm{Vol}(M)W(0) \to \infty$. The existence of $u$, with $E_\varepsilon(u) = c_\varepsilon(p)$ and $\mathrm{Ind}(u) \leq p \leq \mathrm{Ind}(u)+\mathrm{Nul}(u)$ follows from the Palais-Smale condition and Ghoussoub's min-max theorem for cohomological families (see [7]). The same upper bound implies $\limsup_{\varepsilon\to 0} c_\varepsilon(p) < +\infty$. You can prove the lower bound in the following way. First, show that $c_\varepsilon(p) > 0$ (this is easy since the only solutions with energy 0 are the constants $\pm 1$). Finally, notice that $0 < E_\varepsilon(u) < E_\varepsilon(0)$, implies that the solution is not trivial. Then, it has a nodal set from Exercise 6. Deduce a lower bound for $E_\varepsilon(u)$ from Exercise 7. □

## 6. Construction of solutions by minimization and gluing

Even for familiar geometries it is hard to determine how the min-max solutions constructed above will look like. Because of this, sometimes it is useful to construct solutions in other ways. In this section, we discuss how to use Exercises 8 and 9 to construct solutions by gluing methods. With this approach we can have a good picture of how certain solutions look like and also give very good bounds for their energy. On the other hand, it is hard to estimate their Morse index.

**Exercise 14.** *Construct a solution to* (1) *on $S^n$ with nodal set equal to an equator.*

*Hint: Use Exercise 8 to solve the Dirichlet problem on each hemisphere. Choose the positive solution in one and the negative solution in the other. Use Exercise 9 to conclude that both solutions are rotationally symmetric and indeed, their gradient coincides on the equator. Conclude that the value of the solutions, their gradient and Laplacian, coincide on the equator. Deduce from this that when we glue both solutions we obtain a weak solution of* (1) *(and therefore a smooth solution).*

The following exercises deal with similar ideas in different geometries.

**Exercise 15.** *Construct a solution to* (1) *on a rotationally symmetric torus $\mathbb{T}^n$ whose nodal set is exactly two antipodal slices of $\mathbb{T}^{n-1}$ (you can actually do it for any even number of equidistant slices).*

**Exercise 16.** *Construct a solution to* (1) *on $S^n$ whose nodal set is two orthogonal equators.*

*Hint: In this case the nodal set is singular. Similar ideas work for showing that after gluing we obtain a weak solution.*

**Exercise 17.** *Construct a solution to* (1) *on $\mathbb{R}^2$ whose nodal set is two orthogonal lines.*

*Hint: First, construct a solution on a ball of radius $R > 0$ to the Dirichlet problem, having nodal set two orthogonal segments of lines passing through the origin. After this, take the limit as $R \to \infty$ and use standard Schauder estimates.*

**Exercise 18.** *Construct a solution to* (1) *on $\mathbb{R}^2$ whose nodal set is two orthogonal lines.*

*Hint: First, construct a solution on a ball of radius $R > 0$ to the Dirichlet problem, having nodal set two orthogonal segments of lines passing through the origin. After this, take the limit as $R \to \infty$ and use standard Schauder estimates.*

**Exercise 19** (*). *Construct a solution to* (1) *on $S^n$ whose nodal set is two parallels. More precisely, $\{u = 0\} = S^n \cap \{x_{n+1} = \pm s_0\}$, for some $|s_0| < 1$.*

*Hint: Define regions $A_\tau = S^n \cap \{|x_{n+1}| < \tau\}$ and $S^n \setminus A_\tau = D_\tau^+ \cup D_\tau^-$, where $D_\tau^\pm$ are the two disks forming the complement of the annulus $A_\tau$. Glue together the positive solutions with Dirichlet condition on $D_\tau^\pm$ and the negative solution with Dirichlet condition on $A_\tau$. Show that each solution satisfies an homogeneous Neumann condition at the boundary, which varies continuously with respect to $\tau$. Finally, show that when $\varepsilon$ is small enough, there exists $\tau \in [0, 1]$ such that the gradients of the solutions coincide at the boundary of $A_\tau$. This is the solution you are looking for.*

**Exercise 20** (**). *In the example above, you can actually prove that as $\varepsilon \to 0$, the zero level set accumulates on an equator with multiplicity 2.*

*Hint: Show first that the solutions have finite energy. This can be done by expressing the solution in Fermi coordinates around the nodal set and using Exercise 10. In particular by Hutchinson-Tonegawa* [9]*, the energy will accumulate on an stationary varifold. After passing to a subsequence, we can assume that the zero level set converges to two parallels or the equator. Prove that the energy must concentrate on this limit set (again you can use Exercise 10 for this). Since two parallels are not stationary, it follows that it must converge to the equator.*

**Exercise 21.** *Construct a solution to* (1) *on $\mathbb{RP}^n$ whose nodal converges to a copy of $\mathbb{RP}^{n-1}$ with multiplicity 2.*

*Hint: First, show that the solutions on $S^n$ from the previous exercise are even.*

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