SOLUTIONS TO MIDTERM 1
MA203000 - WINTER 2017
Instructor: Marco Guaraco

Please write clearly and show all steps of your reasoning with full mathematical rigor. If you find anything during the test that you think might be a typo please let me know as soon as possible. Partial credit is possible.

Name: Student ID:

Problem	Score
1	/25
2	/25
3	/50
Total	/100

Problem 1. [25pts] (True or false) For each of the following statements mark true or false. If the statement is true no further explanation is needed, if it is false give a counter-example.

(1) The union of convex sets is convex.
(2) If $E \subset \mathbb{R}$ then $\bar{E} = E'$ (i.e. the closure of a set is always equal to the set of its limit points).
(3) A countable intersection of open sets can be closed.
(4) If E is a dense subset of a metric space then E^C is not dense.
(5) If $E \subset \mathbb{R}$ is closed and bounded from above then $\sup E \in E$.

Answer:

(1) False: A union of disjoint convex sets is not convex.
(2) False: Take $E = \{p\}$, then $\bar{E} = \{p\}$ but $E' = \emptyset$.
(3) True: Consider $A_n = (-1/n, 1/n)$. Then $\cap_{n \in \mathbb{N}} A_n = \{0\}$ which is closed.
(4) False: The rationals and irrationals are both dense in $\mathbb{R}$.
(5) True.

Problem 2. [25pts] Prove that there is not rational number x such that $x^2 = 28$.

Answer: Assuming there is a simplified fraction $x = m/n$ such that $x^2 = 28$, we conclude that $m^2 = 4 \cdot 7 \cdot n^2$. Then m^2 is a multiple of 7. Since 7 is a prime number then m is also a multiple of 7. Now m^2 is divisible by 49 and then $4 \cdot 7 \cdot n^2$ is also divisible by 49. In particular, n^2 (and also n) must be divisible by 7, contradicting the fact that m/n is a simplified fraction.

Problem 3. [50pts] The Cauchy-Schwarz inequality for real numbers states that

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \sum_{i=1}^{n} x_i^2 \cdot \sum_{i=1}^{n} y_i^2,$$

for any real numbers $x_i \in \mathbb{R}$ and $y_i \in \mathbb{R}$, with $n \in \mathbb{R}$ and $i \in \{1, 2, \ldots, n\}$.

Use it to prove the following statements:

(1) [10pts] If $a_1, \ldots, a_n$ are positive real numbers then

$$\sum_{i=1}^{n} a_i \sum_{i=1}^{n} \frac{1}{a_i} \geq n^2.$$

(2) [15pts] If $b_1, \ldots, b_n$ are positive real numbers then

$$\frac{a_1^2}{b_1} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + \cdots + a_n)^2}{b_1 + \cdots + b_n}.$$

(3) [15pts] Use (2) to conclude that if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then

$$n\sqrt{n}\|x\|_\infty \geq \sqrt{n}\|x\|_2 \geq \|x\|_1 \geq \|x\|_\infty.$$

(4) [10pts] Argue that (3) implies that the metrics associated with the norms $\|\cdot\|_p$ with $p = 1, 2$ and ∞ define the same open sets.

Answer:

(1) Follows directly taking $x_i = \sqrt{a_i}$ and $y_i = 1/\sqrt{a_i}$, which we can do since a_i are positive.

(2) Similarly, taking $x_i = a_i/\sqrt{b_i}$ and $y_i = \sqrt{b_i}$ gives

$$(b_1 + \cdots + b_n)\left(\frac{a_1^2}{b_1} + \cdots + \frac{a_n^2}{b_n}\right) \geq (a_1^2 + \cdots + a_n)^2.$$

Dividing by $b_1 + \cdots + b_n$ we obtain the result.

(3) Without lost of generallity, we can assume that $max\{|x_1|, \ldots, |x_n|\} = |x_1|$.

The inequality $\|x\|_\infty \leq \|x_1\|$ is exactly $|x_1| \leq |x_1| + \cdots + |x_n|$, which is obviously true.

To see $\sqrt{n}\|x\|_2 \geq \|x_1\|$, take $b_i = 1$ and $a_i = |x_i|$ in (2) i.e.

$$x_1^2 + \cdots + x_n^2 \geq \frac{(|x_1| + \cdots + |x_n|)^2}{n}.$$

The result follows after multiply by n and taking the square root of both sides.

Finally, we need to show $n\sqrt{n}\|x\|_\infty \geq \sqrt{n}\|x\|_2$, which is the same as $n\|x\|_\infty \geq \|x\|_2$. In fact, notice that $\|x\|_\infty^2 \geq x_i^2$, for every $i = 1, \ldots, n$. Adding these inequalities it follows that

$$n\|x\|_\infty^2 \geq x_1^2 + \cdots + x_n^2.$$

The result follows after taking square roots on both sides.

(4) By definition the metrics associated with the norms are $d_p(x,y) = \|x-y\|_p$. (3) implies that for every $p,q \in \{1,2,\infty\}$, there are constants $A > 0$ and $B > 0$ such that

$$A \cdot d_q(x,y) \leq d_p(x,y) \leq B \cdot d_q(x,y).$$

These inequalities imply that every neighboorhood of $x \in \mathbb{R}^n$ on the metric d_p contains a neighboorhood of x with respect to the metric d_q (perhaps with a different radius) , and viceversa. It follows that $A \subset R^n$ is open in the metric d_p if and only if it is open in the metric d_q.