STRICHARTZ ESTIMATES FOR CHARGE TRANSFER MODELS

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Abstract. In this note, we prove Strichartz estimates for scattering states of
calar charge transfer models in \( \mathbb{R}^3 \). Following the idea of Strichartz estimates
based on [3, 10], we also show that the energy of the whole evolution is bounded
independently of time without using the phase space method, as for example,
in [5]. One can easily generalize our arguments to \( \mathbb{R}^n \) for \( n \geq 3 \). We also
discuss the extension of these results to matrix charge transfer models in \( \mathbb{R}^3 \).

1. Introduction

In this note, following the work of [10, 1], charge transfer models for Schrödinger
equations in \( \mathbb{R}^3 \) will be considered. We study the time-dependent charge transfer Hamiltonian

\[
H(t) = -\frac{1}{2} \Delta + \sum_{j=1}^{m} V_j(x - \vec{v}_j t)
\]

with rapidly decaying smooth potentials \( V_j(x) \), say, exponentially decaying and a
set of mutually non-parallel constant velocities \( \vec{v}_j \). Strichartz estimates for the
evolution

\[
\frac{1}{i} \partial_t \psi + H(t) \psi = 0
\]

associated with a charge transfer Hamiltonian \( H(t) \) will be proved.

The starting point is the well-known \( L^p \) estimates for the free Schrödinger equation \( (H_0 = -\frac{1}{2} \Delta) \) on \( \mathbb{R}^n \):

\[
\| e^{iH_0 t} f \|_{L^p} \leq C_p |t|^{-(\frac{n}{2} - \frac{1}{p})} \| f \|_{L^{p'}} ,
\]

where \( 2 \leq p \leq \infty, \frac{1}{p} + \frac{1}{p'} = 1 \).

To analyze the dispersive estimate of linear Schrödinger equations with poten-
tials, we consider the dispersive estimates of the Schrödinger flow

\[
e^{itH} P_c, \quad H = -\frac{1}{2} \Delta + V
\]

on \( \mathbb{R}^n \), where \( P_c \) is the projection onto the continuous spectrum of \( H \). For Schrödinger
equations with potentials, there may be bound states, i.e., \( L^2 \) eigenfunctions of \( H \).
Under the evolution \( e^{itH} \), such bound states are merely multiplied by oscillating
factors and thus do not disperse. So we need to project away any bound state. \( V \) is
a real-valued potential that is assumed to satisfy some decay condition at infinity.

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This decay is typically expressed in terms of the point-wise decay $|V(x)| \leq C \langle x \rangle^{-\beta}$, for all $x \in \mathbb{R}^n$ and for some $\beta > 0$. We use the notation $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$. Occasionally, we will use an integrability condition $V \in L^p(\mathbb{R}^n)$ (or a weighted variant of it) instead of a point-wise condition. These decay conditions will also be such that $H$ is asymptotically complete:

$$L^2(\mathbb{R}^n) = L^2_{p.p.}(\mathbb{R}^n) \oplus L^2_{a.c.}(\mathbb{R}^n)$$

where the spaces on the right-hand side refer to the span of all eigenfunctions, and the absolutely continuous subspace, respectively.

The dispersive estimate for the linear Schrödinger equations with potentials, which we will be most concerned with is of the form

$$\sup_{t \neq 0} |t|^{\frac{n}{2}} \|e^{itH}P_c f\|_{L^\infty_x} \leq C \|f\|_{L^1_x} , \quad \forall f \in L^1_x(\mathbb{R}^n) \cap L^2_x(\mathbb{R}^n).$$

Interpolating with the $L^2$ bound $\|e^{itH}P_c f\|_{L^2_x} \leq C \|f\|_{L^2_x}$, we get

$$\sup_{t \neq 0} |t|^n(\frac{1}{2} - \frac{1}{p}) \|e^{itH}P_c f\|_{L^p_x} \leq C \|f\|_{L^1_x} \quad \forall f \in L^1_x(\mathbb{R}^n) \cap L^2_x(\mathbb{R}^n),$$

where $1 \leq p \leq 2$.

It is well-known that via a $T^*T$ argument the dispersive estimate (5) gives rise to the class of Strichartz estimates

$$\|e^{itH}P_c f\|_{L^q_tL^p_x} \lesssim \|f\|_{L^2}$$

for all $\frac{2}{q} + \frac{n}{p} = \frac{n}{2}$. The endpoint $q = 2$ holds for $n \geq 3$ but it is not captured by this approach, see [7].

Roughly speaking, Strichartz estimates can be regarded as smoothing effects in $L^p_x$ spaces. For example, when we consider the free Schrödinger equation, compared with the trivial conservation of the $L^2$ norm of the solution, in Strichartz estimates one gains space integrability from $p = 2$ to $p > 2$, if one chooses to loses time integrability from $q = \infty$ to $q < \infty$. In other words, we gain space integrability in the integral sense. To be more precise, we can take a function $g \in L^2$ with $g \notin L^p_x$ for $p > 2$. If we take $f = e^{it\Delta}g$ as the initial data for the free linear Schrödinger equation, then we can see that at $t = t_0$, $e^{-it_0\Delta}f \notin L^p_x$. So without integration or averaging in time, there is no hope to get $L^p_x$ estimates for all times for general $L^2$ initial data. Strichartz estimates are crucial for the study of long-time behavior of associated nonlinear models.

For the results and historical progress of dispersive estimates and smoothing effects of Schrödinger operators, one can find further details and references in [11].

There are extra difficulties for Schrödinger equations with time-dependent potentials. For example, given a general time-dependent potential $V(x,t)$, it is not clear how to introduce an analog of bound states and the spectral projection. And the evolution of equation might not satisfy group properties any more. In this paper, we focus on a particular case of time-dependent potentials, i.e. the charge transfer models in $\mathbb{R}^3$.

Firstly, we consider the scalar model in the following sense:
Definition 1.1. By a charge transfer model we mean a Schrödinger equation

\[ \frac{1}{i} \partial_t \psi - \frac{1}{2} \Delta \psi + \sum_{j=1}^{m} V_j (x - \vec{v}_j t) \psi = 0, \]

\[ \psi|_{t=0} = \psi_0, \ x \in \mathbb{R}^3. \]

where \( \vec{v}_j \)'s are distinct vectors in \( \mathbb{R}^3 \), and the real potentials \( V_k \) are such that

1) \( V_k \) is time-independent and decays exponentially (or has compact support)
2) 0 is neither an eigenvalue nor a resonance of the operators

\[ H_k = -\frac{1}{2} \Delta + V_k(x). \]

Recall that \( \psi \) is a resonance at 0 if it is a distributional solution of the equation \( H_k \psi = 0 \) which belongs to the space \( \{ f : \langle x \rangle^{-\sigma} f \in L^2 \} \) for any \( \sigma > \frac{1}{2} \), but not for \( \sigma = 0 \).

To simplify our argument, we discuss when \( m = 2 \) case with \( V_1 \) is stationary and \( V_2 \) moves along \( \vec{e}_1 \) with the unit speed. It is easy to see our arguments work for general cases.

Remark. The assumptions are always assumed when we want to prove dispersive estimate and Strichartz estimates, e.g. [6, 11, 12, 10, 1]. The decay required of the potentials is not optimal but merely for convenience.

An indispensable tool in the study of charge transfer models are the Galilei transformations

\[ g_{\vec{v},y}(t) = e^{i|\vec{v}|^2/2} e^{ix \cdot \vec{v}} e^{-i(y+\vec{v}t) \cdot \vec{p}}, \]

cf. [5, 1, 10], where \( \vec{p} = -i\vec{\nabla} \). They are the quantum analogues of the classical Galilei transforms

\[ x \mapsto x - t\vec{v} - y, \quad \vec{p} \mapsto \vec{p} - \vec{v}. \]

To see this, we take a Schwartz function \( f \) such that \( \hat{f} \) is centered around \( \vec{v} \), then \( g_{\vec{v},y}(t)f \) is centered around \( t\vec{v} + y \) and \( \hat{g}_{\vec{v},y}(t)f \) is centered around \( \vec{v} \). The Galilei transformations have a very important conjugacy property:

\[ g_{\vec{v},y}(t) e^{it \hat{H}} = e^{it \hat{H}} g_{\vec{v},y}(0). \]

Moreover, notice that with \( H = -\frac{1}{2} \Delta + V \), then

\[ \psi(t) := g_{\vec{v},y}(t)^{-1} e^{-itH} g_{\vec{v},y}(0) \psi_0, \quad g_{\vec{v},y}(t)^{-1} = e^{-iy \vec{v}} g_{-\vec{v},-y}(t), \]

solves

\[ \frac{1}{i} \partial_t \psi - \frac{1}{2} \Delta \psi + V (\cdot - t\vec{v} - y) \psi = 0, \quad \psi|_{t=0} = \psi_0. \]

Another important property of the Galilei transformations is that \( g_{\vec{v},y}(t) \) are isometries in all \( L^p \) spaces. Finally, in our case, as discussed above, we always assume \( y = 0 \). To simplify our notations, we write \( \xi(t) := g_{\vec{e}_1,0}(t) \) and notice that \( \xi(\vec{e}_1)^{-1} = g_{-\vec{e}_1,0}(t) \).

We recall some consequences from [10, 1]. Again, we consider

\[ \frac{1}{i} \partial_t \psi - \frac{1}{2} \Delta \psi + V_1 \psi + V_2 (\cdot - t\vec{e}_1) \psi = 0, \quad \psi|_{t=0} = \psi_0. \]
with $V_1$ and $V_2$ decaying rapidly. Let $w_1, \ldots, w_m$ and $u_1, \ldots, u_{\ell}$ be the normalized bound states of $H_1$ and $H_2$ associated to the negative eigenvalues $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_{\ell}$ respectively (notice that by our assumptions, 0 is not an eigenvalue).

Following the notations in [10], we denote by $P_b(H_1)$ and $P_b(H_2)$ the projections onto the the bound states of $H_1$ and $H_2$, respectively, and let $P_c(H_i) = Id - P_b(H_i), i = 1, 2$. To be more explicit, we have

$$P_b(H_1) = \sum_{i=1}^{m} \langle \cdot, w_i \rangle w_i, \quad P_b(H_2) = \sum_{j=1}^{\ell} \langle \cdot, u_j \rangle u_j. \tag{16}$$

It is well-known, from the standard case with stationary potentials that we need to project away from bound states as we discussed at the very beginning. Here following [10], we recall the analogous condition in our case.

**Definition 1.2.** Let $U(t, 0)\psi_0 = \psi(t, x)$ be the solution of equation (15). We say that $\psi_0$ or $\psi(x, t)$ is asymptotically orthogonal to the bound states of $H_1$ and $H_2$ if

$$\|P_b(H_1) U(t, 0)\psi_0\|_{L^2} + \|P_b(H_2, t) U(t, 0)\psi_0\|_{L^2} \to 0, \quad t \to \pm \infty. \tag{17}$$

Here

$$P_b(H_2, t) := g_{\sigma_1}(t)^{-1} P_b(H_2) g_{\sigma_1}(t), \quad \forall t.$$ 

It is clear that all $\psi_0$ that satisfy (17) form a closed subspace of $L^2(\mathbb{R}^n)$. We call elements in this subspace scattering states at $t = 0$ and denote the subspace by $H_s(0)$. We name $H_s(0)$ as scattering space at $t = 0$. With $H_s(0)$, we define $P_s(0)$ to be the projection onto $H_s(0)$.

**Remark.** The subspace above coincides with the space of scattering states for the charge transfer problem which appears in Graf’s asymptotic completeness result [5]. We will see more details in Section 2.

We now formulate our main results.

**Theorem 1.3** (Strichartz estimates). Consider the charge transfer model as in Definition 1.1 with two potentials in $\mathbb{R}^3$ as above. Suppose the initial data $\psi_0 \in L^2(\mathbb{R}^3)$ is asymptotically orthogonal to the bound states of $H_1$ and $H_2$ in the sense of Definition 1.2. Then for $\psi(t, x) = U(t, 0)\psi_0$ and a Schrödinger admissible pair $(p, q)$ in $\mathbb{R}^3$, i.e.,

$$\frac{2}{p} + \frac{3}{q} = \frac{3}{2}, \tag{18}$$

with $2 \leq q \leq \infty, p \geq 2$, we have

$$\|\psi\|_{L^p_t((0, \infty), L^q)} \leq C \|\psi_0\|_{L^2}. \tag{19}$$

We also have the boundedness of the energy.

**Theorem 1.4.** Let $\psi_0 \in H^1(\mathbb{R}^3)$ and $\psi(t, x) = U(t, 0)\psi_0$ be a solution to (15) with the initial data $\psi_0$. Then

$$\sup_{t \in \mathbb{R}} \|U(t, 0)\psi_0\|_{H^1} \leq C \|\psi_0\|_{H^1}. \tag{20}$$

The paper is organized as follows: In Section 2, we will recall some results from [10, 1]. Then in Section 3, we establish Strichartz estimates for the evolution that is not associated to the bound states of $H_j$ for the scalar charge transfer model. In Section 4, we will show the energy of the whole evolution is bounded independently of time. Finally, we will generalize our arguments to non-selfadjoint matrix cases in Section 5.
2. Preliminaries

In this section, we formulate the important results from [10, 1] which are crucial for later sections.

First of all, if the evolution is asymptotically orthogonal to the bound states of $H_1$ and $H_2$, we can actually get a decay rate for

$$
\| P_b(H_1)U(t,0)\psi_0\|_{L^2} + \| P_b(H_2, t)U(t,0)\psi_0\|_{L^2} \to 0.
$$

Proposition 2.1 ([10], Proposition 3.1). Let $\psi(t, x)$ be a solution to (15) which is asymptotically orthogonal to the bound states of $H_1$ and $H_2$ in the sense of Definition 1.2. Then we have the decay rate that

$$
(21) \quad \| P_b(H_1)U(t,0)\psi_0\|_{L^2} + \| P_b(H_2, t)U(t,0)\psi_0\|_{L^2} \lesssim e^{-\alpha|t|} \| \psi_0\|_{L^2}
$$

for some $\alpha > 0$.

As pointed out above, $\forall \psi_0 \in L^2$ such that the asymptotically orthogonal condition 17 are satisfied, they form a subspace $H_s(0) \subset L^2$. We can do a more general time-dependent construction. Denote the evolution starting from $\tau$ to $t$ by $U(t, \tau)$. Similar as our original construction there is a subspace $H_s(\tau) \subset L^2$ such that for $\psi \in H_s(\tau)$,

$$
\| P_b(H_1)U(t, \tau)\psi\|_{L^2} + \| P_b(H_2, t)U(t, \tau)\psi\|_{L^2} \to 0.
$$

Similarly as above, we can also obtain a decay rate that for some $\alpha > 0$,

$$
\| P_b(H_1)U(t, \tau)\psi\|_{L^2} + \| P_b(H_2, t)U(t, \tau)\psi\|_{L^2} \lesssim e^{\alpha(t-\tau)} \| \psi\|_{L^2}, \, \psi \in H_s(s).
$$

It is crucial to notice an important property of $H_s(\tau)$.

Lemma 2.2. Denote $P_s(\tau)$ to be the projection onto $H_s(\tau)$. Then for arbitrary $s, \tau \in \mathbb{R}$,

$$
(22) \quad P_s(\tau)U(s, \tau) = U(s, \tau)P_s(\tau).
$$

Proof. Notice that for $\psi \in H_s(\tau)$, then $U(s, \tau)\psi \in H_s(s)$. Since

$$
\| P_b(H_1)U(t, s)U(s, \tau)\psi\|_{L^2} + \| P_b(H_2, t)U(t, s)U(s, \tau)\psi\|_{L^2}
$$

$$
= \| P_b(H_1)U(t, \tau)\psi\|_{L^2} + \| P_b(H_2, t)U(t, \tau)\psi\|_{L^2} \to 0
$$

as $t \to \infty$ by the definition of $H_s(\tau)$. Then again by the definition of $H_s(s)$, it is clear $U(s, \tau)\psi \in H_s(s)$. Conversely, by symmetry, for $\psi \in H_s(s)$, then $U(\tau, s)\psi \in H_s(\tau)$. Therefore, we have the scattering spaces are invariant under the flow $U(s, \tau)$,

$$
(23) \quad H_s(s) = U(s, \tau)H_s(\tau).
$$

Let $\phi \in L^2$, then $U(s, \tau)P_s(\tau)\phi \in H_s(s)$ by construction. Then

$$
U(s, \tau)P_s(\tau)\phi = (1 - P_s(s))U(s, \tau)P_s(\tau)\phi + P_s(s)U(s, \tau)P_s(\tau)\phi
$$

$$
= P_s(s)U(s, \tau)P_s(\tau)\phi.
$$

Similarly,

$$
P_s(s)U(s, \tau) = U(s, \tau)P_s(\tau),
$$

as claimed. \qed

If the evolution is asymptotically orthogonal to the bound states, one also have the usual $L^1 \to L^\infty$ dispersive estimate.
Theorem 2.3 ([10],[1]). Consider the charge transfer model as in Definition 1.1 with two potentials as above. Assume $\hat{V}_1, \hat{V}_2 \in L^1(\mathbb{R}^n)$. Then for any initial data $\psi_0 \in L^1(\mathbb{R}^n)$, which is asymptotically orthogonal to the bound states of $H_1$ and $H_2$ in the sense of Definition 1.2, one has the decay estimate

$$\|U(t,0)\psi_0\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}} \|\psi_0\|_{L^1}.$$  

A similar estimate holds for any number of potentials.

Note that since the potentials depend on time, Strichartz estimates do not follow from the dispersive estimate and $TT^*$ argument.

With the decay estimate (24), we obtain the asymptotic completeness of the charge transfer Hamiltonian:

Theorem 2.4 ([10, 1]). Let $w_1, \ldots, w_m$ and $u_1, \ldots, u_\ell$ be the normalized bound states of $H_1$ and $H_2$ associated to the negative eigenvalues $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_\ell$. Then for any initial data $\psi_0 \in L^2(\mathbb{R}^n)$, the solution $\psi(t,x) = U(t,0)\psi_0$ of the charge transfer model, equation (15), can be written in the form

$$\psi(t,x) = U(t,0)\psi_0 = \sum_{r=1}^m A_re^{-i\lambda_r t}w_r + \sum_{s=1}^\ell B_se^{-i\mu_s t}g_{-\ell_1}(t)u_s + e^{-it\frac{\Delta}{2}}\phi_0 + R(t)$$

with some choice of the constants $A_r$, $B_s$ and the function $\phi_0$. The remainder term $R(t)$ satisfies the estimate

$$\|R(t)\|_{L^2} \to 0, \ t \to \pm \infty.$$  

With the asymptotic completeness of the charge transfer Hamiltonian, we can construct a time-dependent decomposition of $L^2$ with scattering states and analogous of bound states associated with $H_1$ and $H_2$. The construction should be similar as [5, 13]. Following the notations in [10, 5], with the proof of Theorem 2.4, we know the existence of the following wave operators in $L^2$: for $s \in \mathbb{R}$,

$$\Omega_0^-(s) = \lim_{t \to \infty} U(s,t)e^{-iH_0(t-s)}$$

with limits are taken as strong operator topology. From [10], the ranges of the above operators has the following relation:

$$L^2 = \text{Ran}\Omega_0^-(s) \oplus \text{Ran}\Omega_1^-(s) \oplus \text{Ran}\Omega_2^-(s).$$

Naturally the above constructions will introduce a time-dependent decomposition of $L^2$ and one can observe that

$$\text{Ran}\Omega_0^-(\tau) = H_s(\tau).$$

We introduce projections $P_i(\tau)$ to the projection onto the range of $\Omega_i^{-}(\tau)$, $i = 1, 2$. Clearly. $\text{Ran}\Omega_1^-(\tau)$ and $\text{Ran}\Omega_2^-(\tau)$ are analogous as the spans bound states associated with $H_1$ and $H_2$ respectively. Notice that by construction, one can find a basis for $\Omega_i^{-}(\tau)$, $i = 1, 2$. With our notations above, $w_1, \ldots, w_m$ and $u_1, \ldots, u_\ell$ be the normalized bound states of $H_1$ and $H_2$ associated to the negative eigenvalues $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_\ell$ respectively. Then \( \{w_{i,t} = \Omega_1^{-}(\tau)w_i\}_{i=1}^m \) is a basis for
RanΩ−1(τ). Similarly, \( \{ u_{j,\tau} = \Omega_2^{-1}(\tau)g(c_j(\tau))u_j \}_{j=1}^{m} \) is a basis for RanΩ2(τ). By asymptotic completeness and intuition, as \( \tau \to \infty \), \( \Omega_1^{-1}(\tau)w_i \to w_i \) and \( \Omega_2^{-1}(\tau)g(\tau)u_j \to g(c_1(\tau))^{-1}u_j \). Actually, following [10], one can actually extract a convergent rate. We focus on \( \Omega_1^{-1}(\tau)w_i \to w_i \) and for the other case, we just need to apply the same argument after applying a Galilei transformation.

**Proposition 2.5.** For some \( \alpha > 0 \),

\[
\| \Omega_1^{-1}(\tau)w_i - w_i \|_{L^2} \lesssim e^{-\alpha \tau}.
\]

**Proof.** For \( 1 \leq i \leq m \), with Duhamel’s formula

\[
U(s,t)e^{-iH_1(t-s)}w_i = w_i + i \int_s^t U(s,t)V_2(\cdot - c_1^2(t))e^{-i\lambda_1(\tau-s)}w_i d\tau.
\]

It suffices to estimate the \( L^2 \) norm of

\[
\int_s^\infty U(s,t)V_2(\cdot - c_1^2(t))e^{-i\lambda_1(\tau-s)}w_i d\tau.
\]

By Agmon’s estimate,

\[
\left\| \int_s^\infty U(s,t)V_2(\cdot - c_1^2(t))e^{-i\lambda_1(\tau-s)}w_i d\tau \right\|_{L^2} \lesssim \left\| \int_s^\infty V_2(\cdot - c_1^2(t))w_i d\tau \right\|_{L^2} \lesssim e^{-\alpha s}.
\]

Therefore

\[
\| \Omega_1^{-1}(\tau)w - w_i \|_{L^2} \lesssim e^{-\alpha \tau},
\]

as claimed. \( \square \)

3. Strichartz estimates

In this section, we prove Strichartz estimates for charge transfer models. The ideas will be based on methods in [3, 10]. Certainly, we need to project away from the bound states of \( H_1 \) and the moving bound states associated to \( H_2(t) \). We will show certain weighted estimates for the evolution of states in the scattering space defined in [10] and in the sense of Definition 1.2.

Now we formulate the following two estimates when our initial state is in the scattering space. The first one is:

**Lemma 3.1.** For \( \sigma > \frac{3}{2} \) and \( t \geq t_0 \)

\[
\| (x - x_0)^{-\sigma}U(t,t_0)P_s(t_0)(x - x_1)^{-\sigma} \|_{2 \to 2} \leq C \frac{1}{(t-t_0)^{\frac{3}{2}}}
\]

for all \( x_0 \) and \( x_1 \).

Here \( 2 \to 2 \) means the norm as an operator from \( L^2 \) to \( L^2 \) and \( P_s \) defined as the projection onto the scattering space as above in sense of Definition 1.2. Also as usual, \( \langle x \rangle = (|x|^2 + 1)^{\frac{1}{2}} \). The second weighted estimate we want to show is the following:

**Lemma 3.2.** For \( \sigma > \frac{3}{2} \)

\[
\int_0^\infty \left\| (x - x(t))^{-\sigma}U(t,t_0)P_s(t_0)u \right\|_{L^2_x}^2 dt \leq C \| u \|_{L^2_x}^2
\]

for all \( x(t) \in C([0,\infty),\mathbb{R}^3) \).
Heuristically, we can see the above two estimates hold for the evolution of a free Schrödinger equation since a free particle moves towards infinity. The weights just play roles like indicator functions of certain finite regions. Then surely, as time evolves, the particle will leave any of those regions. So we have the decay of the wave function. In our case, the state in the scattering space will just move asymptotically like a free particle, so we should expect the above result. The second estimate is a variant of the above heuristics adjusted to our model since we have moving potentials.

Before we prove Lemma 3.1 and Lemma 3.2, we show how to derive Strichartz estimates for the charge transfer model based on them.

**Proof of Theorem 1.3.** Let \( \psi(t) = U(t, 0)\psi_0 \) and by our assumption we have \( P_s(0)\psi_0 = \psi_0 \). Rewrite the charge transfer model as

\[
i\psi_t + \frac{1}{2} \Delta \psi = V_1 \psi + V_2(\cdot - te_1)\psi.
\]

Now we apply the endpoint Strichartz estimate [7] for the free Schrödinger equation, we get for a Schrödinger admissible pair \((p, q)\) in \( \mathbb{R}^3 \), one has

\[
\|\psi\|_{L_t^p([0,\infty), L_x^q)} \leq C \|V_1 \psi + V_2(\cdot - te_1)\psi\|_{L_t^p([0,\infty), L_x^q)} + \|\psi_0\|_{L_x^q}
\]

Since our potentials decay fast, we can pick \( m \) large (in particular \( m > \frac{3}{2} \)) such that by Hölder’s inequality we have,

\[
\|V_1 \psi + V_2(\cdot - te_1)\psi\|_{L_t^p([0,\infty), L_x^q)} \leq \|V_1 \psi\|_{L_t^p([0,\infty), L_x^{q/2})} + \|V_2(\cdot - te_1)\psi\|_{L_t^p([0,\infty), L_x^{q/2})}
\]

\[
\|V_1 \psi\|_{L_t^p([0,\infty), L_x^{q/2})} \leq CV \|\langle x \rangle^{-m} \psi\|_{L_t^p([0,\infty), L_x^{q/2})}
\]

\[
\|V_2(\cdot - te_1)\psi\|_{L_t^p([0,\infty), L_x^{q/2})} \leq CV \|\langle x - te_1 \rangle^{-m} \psi\|_{L_t^p([0,\infty), L_x^{q/2})}
\]

Now by our above two claimed estimates Lemma 3.1 and Lemma 3.2, we have

\[
\|\langle x \rangle^{-m} \psi\|_{L_t^p([0,\infty), L_x^q)} \leq C \|\psi_0\|_{L^2},
\]

\[
\|\langle x - te_1 \rangle^{-m} \psi\|_{L_t^p([0,\infty), L_x^q)} \leq C \|\psi_0\|_{L^2}.
\]

Then combine all estimates above, we get

\[
\|\psi\|_{L_t^p([0,\infty), L_x^q)} \leq C \|\langle x \rangle^{-m} \psi\|_{L_t^p([0,\infty), L_x^q)} + C \|\langle x - te_1 \rangle^{-m} \psi\|_{L_t^p([0,\infty), L_x^q)} \leq C \|\psi_0\|_{L^2}.
\]

Therefore, we have the desired Strichartz estimate

\[
\|\psi\|_{L_t^p([0,\infty), L_x^q)} \leq C \|\psi_0\|_{L^2}.
\]
as claimed. \( \square \)

In the next section, we will show as a byproduct of Strichartz estimates, (19), we can get the energy boundedness of the whole evolution of the charge transfer model.
3.1. Proof of lemmas 3.1 and 3.2. To rigorously show Lemmas 3.1 and 3.2 are consistent with our heuristics, we consider the free evolution first. We claim that the first estimate (34) holds for the free Schrödinger equation.

Lemma 3.3. For $\sigma > \frac{3}{2}$

$$\left\| (x - x_0)^{-\sigma} e^{i \hat{\Delta} (t-t_0)} (x - x_1)^{-\sigma} \right\|_{2 \to 2} \leq C \frac{1}{(t-t_0)^{\frac{3}{2}}}$$

for all $x_0$ and $x_1$ in $\mathbb{R}^3$.

Proof. Let $s = t - t_0$, then if $|s| \leq 1$, clearly by $\left\| e^{i \hat{\Delta} s} \right\|_{2 \to 2} \leq 1$ and $\sigma > 0$, we can get the desired result.

If $|s| \geq 1$, we apply the dispersive estimate for the free evolution. Then by Young’s inequality we get

$$\left\| (x - x_0)^{-\sigma} e^{i \hat{\Delta} (t-t_0)} (x - x_1)^{-\sigma} \right\|_{2 \to 2} \leq \left\| (x)^{-\sigma} \right\|_{L^2} \left\| e^{is \hat{\Delta}} \right\|_{1 \to \infty} \lesssim |s|^{-\frac{3}{2}}.$$

So the desired estimate holds.

Also the second estimate (35) holds for the free Schrödinger evolution by the endpoint Strichartz estimate, estimate (7).

Lemma 3.4. For $\sigma > \frac{3}{2}$

$$\int_0^\infty \left\| (x - x(t))^{-\sigma} e^{it \hat{\Delta}} u \right\|_{L^2}^2 dt \leq C \| u \|_{L^2}^2.$$

Proof. By Hölder’s inequality, we have

$$\left\| (x - x(t))^{-\sigma} e^{it \hat{\Delta}} u \right\|_{L^2_x} \lesssim \left\| e^{it \hat{\Delta}} u \right\|_{L^5_x}.$$

Then by the endpoint Strichartz estimate in $\mathbb{R}^3$, [7], we have

$$\left\| e^{it \hat{\Delta}} u \right\|_{L^7_x L^5_t} \leq C \| u \|_{L^2_x}.$$

Therefore, we can conclude

$$\int_0^\infty \left\| (x - x(t))^{-\sigma} e^{it \hat{\Delta}} u \right\|_{L^2}^2 dt \leq C \| u \|_{L^2}^2.$$

The Lemma is proved.

Now we show Lemmas 3.1 and 3.2 by a bootstrap argument similar to the one in [10]. As usual, the constant $C$ varies from line to line.

First of all, we note the following simple facts: Since $P_s(t_0)u$ satisfies following estimates, for $p \geq 2$ and $t \geq t_0$

$$\| P_b (H_1) U(t, t_0) P_s(t_0) \|_{L^2 \to L^p} \lesssim e^{-\alpha(p)|t|},$$

and

$$\| P_b (H_2, t) U(t, t_0) P_s(t_0) \|_{L^2 \to L^p} \lesssim e^{-\beta(p)|t|}.$$

Then surely,

$$\left\| (x - x_0)^{-\sigma} P_b (H_1) U(t, t_0) P_s(t_0) (x - x_1)^{-\sigma} \right\|_{2 \to 2} \leq C \frac{1}{(t-t_0)^{\frac{3}{2}}},$$

$$\left\| (x - x_0)^{-\sigma} P_b (H_2, t) U(t, t_0) P_s(t_0) (x - x_1)^{-\sigma} \right\|_{2 \to 2} \leq C \frac{1}{(t-t_0)^{\frac{3}{2}}}.$$
For the second weighted estimate, with some \( p \geq 2, \)
\[
\left\| (x - x(t))^{-\sigma} P_b(H_1) U(t, t_0) P_s(t_0) u \right\|_{L^2_t L^2_x} \lesssim e^{-\alpha(p) |t|} \| u \|_{L^2_x} ,
\]
\[
\left\| (x - x(t))^{-\sigma} P_b(H_2, t) U(t, t_0) P_s(t_0) u \right\|_{L^2_t L^2_x} \lesssim e^{-\beta(p) |t|} \| u \|_{L^2_x} .
\]

So
\[
\int_0^\infty \left\| (x - x(t))^{-\sigma} P_b(H_1) U(t, t_0) P_s(t_0) u \right\|_{L^2_t L^2_x} dt \lesssim \| u \|_{L^2_x} ^2 ,
\]
and
\[
\int_0^\infty \left\| (x - x(t))^{-\sigma} P_b(H_2, t) U(t, t_0) P_s(t_0) u \right\|_{L^2_t L^2_x} dt \lesssim \| u \|_{L^2_x} ^2 .
\]

By the Duhamel formula, we write
\[
U(t, t_0) P_s(t_0) = e^{i \frac{t}{2} \Delta(t-t_0)} P_s(t_0) + i \int_{t_0}^t e^{i \frac{s}{2} \Delta(t-s)} V_1 U(s, t_0) P_s(t_0) ds
\]
\[+ i \int_{t_0}^t e^{i \frac{s}{2} \Delta(t-s)} V_2 (\cdot - s e_1^\sigma) U(s, t_0) P_s(t_0) ds ,
\]
and let
\[
U(t, t_0) = F + iL + iG
\]
Surely, there is no problem with the free piece \( F \) as we discussed above by Lemmas 3.3 and 3.4.

Now fix \( T \) large enough and apply Gronwall’s equality. Then we can find a large constant \( C(T) \) such that
\[
\left\| (x - x_0)^{-\sigma} U(t, t_0) P_s(t_0) (x - x_1)^{-\sigma} \right\|_{L^2_x} \lesssim C(T) \frac{1}{\langle t - t_0 \rangle^{\frac{1}{2}}}
\]
(40)
\[
\int_0^T \left\| (x - x(t))^{-\sigma} U(t, t_0) P_s(t_0) u \right\|_{L^2} ^2 dt \leq C^2(T) \| u \|_{L^2_x} ^2
\]
(41)

hold for \( t_0 \leq t \leq T \).

Next we imitate the bootstrap process in [10] and [3]. Fix a large constant \( A \) to be determined later. We also assume \( T - t_0 \gg A \). As in [10], for \( t \leq T \) we consider the decomposition of interval \([t_0, t] = [t_0, t_0 + A] \cup [t_0 + A, t - A] \cup [t - A, t]. \) Set
\[
L_1 = \int_{t_0}^{t_0 + A} e^{i \frac{s}{2} \Delta(t-s)} V_1 U(s, t_0) P_s(t_0) ds
\]
\[
L_2 = \int_{t_0 + A}^{t - A} e^{i \frac{s}{2} \Delta(t-s)} V_1 U(s, t_0) P_s(t_0) ds
\]
\[
L_3 = \int_{t - A}^{t} e^{i \frac{s}{2} \Delta(t-s)} V_1 U(s, t_0) P_s(t_0) ds
\]
\[
G_1 = \int_{t_0}^{t_0 + A} e^{i \frac{s}{2} \Delta(t-s)} V_2 (\cdot - s e_1^\sigma) U(s, t_0) P_s(t_0) ds
\]
\[
G_2 = \int_{t_0 + A}^{t - A} e^{i \frac{s}{2} \Delta(t-s)} V_2 (\cdot - s e_1^\sigma) U(s, t_0) P_s(t_0) ds
\]
\[
G_3 = \int_{t - A}^{t} e^{i \frac{s}{2} \Delta(t-s)} V_2 (\cdot - s e_1^\sigma) U(s, t_0) P_s(t_0) ds .
\]
First, we bound $L_1$. With Lemma 3.3, we have
\[
\| \langle x - x_0 \rangle^{-\sigma} L_1 (x - x_1)^{-\sigma} \|_{2 \to 2} \leq C \int_{t_0}^{t_0+A} \frac{1}{(t-s)^{\frac{3}{2}}} \left( \| \langle x \rangle^{-\sigma} U(s,t_0)P_s(t_0)(x-x_1)^{-\sigma} \|_{2 \to 2} \right) ds \\
\leq C(A) \int_{t_0}^{t_0+A} \frac{1}{(t-s)^{\frac{3}{2}} (s-t_0)^{\frac{3}{2}}} ds \\
\leq C(A) \frac{1}{(t-t_0)^{\frac{3}{2}}}.
\]
Here we just emphasize that the constant in above estimate does not depend on $T$.

For the second part, with Lemma 3.4 and bootstrap assumption (41), we get
\[
\int_0^T \| \langle x - x(t) \rangle^{-\sigma} L_2 u(t) \|_{L_x^2}^2 dt \leq C \int_0^T \left( \int_{t_0}^{t_0+A} \frac{1}{(t-s)^{\frac{3}{2}}} \| \langle x \rangle^{-\sigma} U(s,t_0)P_s(t_0)u \|_{L_x^2} \right) ds \\
\leq C \| \langle x \rangle^{-\sigma} U(s,t_0)P_s(t_0)u \|_{L_x^2((t_0,t_0+A),L_x^2)}^2 \\
\leq C(A) \| u \|_{L_x^2}^2.
\]
Also $G_1$ can be bounded similarly.

Next, we analyze $L_2$. With Lemma 3.3 and the bootstrap assumption (40),
\[
\| \langle x - x_0 \rangle^{-\sigma} L_2 (x - x_1)^{-\sigma} \|_{2 \to 2} \leq C \int_{t_0+A}^{t_0+1} \frac{1}{(t-s)^{\frac{3}{2}}} \left( \| \langle x \rangle^{-\sigma} U(s,t_0)P_s(t_0)(x-x_1)^{-\sigma} \|_{2 \to 2} \right) ds \\
\leq C(T) \int_{t_0+A}^{t_0+1} \frac{1}{(t-s)^{\frac{3}{2}} (s-t_0)^{\frac{3}{2}}} ds \\
\leq CC(T)A^{-\frac{3}{2}} \langle t - t_0 \rangle^{-\frac{3}{2}}
\]
for an absolute constant $C$.

For the other estimate, with Lemma 3.4 and bootstrap assumption (41), we conclude that
\[
\int_0^T \| \langle x - x(t) \rangle^{-\sigma} L_2 u(t) \|_{L_x^2}^2 dt \leq C \int_0^T \left( \int_{t_0}^{t_0+A} \langle t - s \rangle^{-\frac{3}{2}} \| \langle x \rangle^{-\sigma} U(s,t_0)P_s(t_0)u \|_{L_x^2} \right) ds \\
\leq h(A) \| \langle x \rangle^{-\sigma} U(s,t_0)P_s(t_0)u \|_{L_x^2((0,T),L_x^2)}^2 \\
\leq h(A)C^2(T) \| u \|_{L_x^2}^2
\]
where
\[
h(A) \lesssim A^{-1}
\]
by Young’s inequality applied to the convolution
\[
\int_{t_0+A}^{t_0+1} \langle t - s \rangle^{-\frac{3}{2}} \| \langle x \rangle^{-\sigma} U(s,t_0)P_s(t_0)u \|_{L_x^2} ds.
\]
So when $A$ is large, we recapture our bootstrap argument conditions, i.e., $h(A)C^2(T)$ will be a small portion of $C(T)$ provided $A$ is large enough. Similar estimates hold for $G_2$.

It remains to analyze $L_3$ and $G_3$. We will expand $U$ again. And the following two versions of weighted estimates for Schrödinger equations with rapidly decaying potentials will be used.

**Lemma 3.5.** For $\sigma > \frac{3}{2}$, and $H_j = -\frac{1}{2}\Delta + V_j$, where $V_j$ satisfies the decay assumption for our charge transfer Hamiltonian, then we have

\begin{equation}
\left\| \langle x - x_0 \rangle^{-\sigma} e^{iH_j(t-t_0)} P_c(H_j) \langle x - x_1 \rangle^{-\sigma} \right\|_{L^2} \leq C \frac{1}{(t - t_0)^{\frac{3}{2}}}
\end{equation}

and

\begin{equation}
\int_0^\infty \left\| \langle x - x(t) \rangle^{-\sigma} e^{itH_j} P_c(H_j) u \right\|_{L^2}^2 dt \leq C \| u \|_{L^2}^2,
\end{equation}

where $P_c(H_j)$ is the projection onto the continuous spectrum of $H_j$.

**Proof.** These two estimates follow from the boundedness of wave operators [12] and Lemma 3.3 and Lemma 3.4. Or one can apply the dispersive estimate and Strichartz estimates for perturbed Schrödinger equations. □

Now we analyze

$$L_3 = \int_{t-A}^t e^{it\frac{1}{2}(t-s)} V_1 U(s,t_0) P_s(t_0) ds.$$  

Splitting $L_3$ with respect to the spectrum of $H_1$, one has

\begin{align*}
L_3 &= \int_{t-A}^t e^{it\frac{1}{2}(t-s)} V_1 P_c(H_1) U(s,t_0) P_s(t_0) ds \\
&\quad + \int_{t-A}^t e^{it\frac{1}{2}(t-s)} V_1 P_b(H_1) U(s,t_0) P_s(t_0) ds \\
&= L_{3,c} + L_{3,b}.
\end{align*}

Surely, there is no problem with $L_{3,b}$ by the discussion at the very beginning of this section $P_b(H_1) U(s,t_0) P_s(t_0)$ decays exponentially.

For $L_{3,c}$, we use the ideas from [10] to decompose our evolution into low velocity and high velocity pieces. For the low velocity piece, we directly use a commutator argument, non-stationary phase and the fact the supports of $V_1$ and $V_2$ become almost disjoint. For the high velocity part, we use a version of the Kato smoothing estimate.

Expanding $U$ with respect to $H_1$, we can write

$$U(t,t_0) = e^{-iH_1(t-t_0)} + i \int_{t_0}^t e^{-iH_1(t-s)} V_2(-s e_1^1) U(s,t_0) ds.$$
Then we can write

\[ L_{3,c} = \int_{t-A}^{t} e^{\frac{i}{2}\Delta(t-s)}V_1P_c(H_1)U(s,t_0)P_s(t_0)\,ds \]

\[ = \int_{t-A}^{t} e^{\frac{i}{2}\Delta(t-s)}V_1P_c(H_1)e^{-iH_1(s-t_0)}P_s(t_0)\,ds \]

\[ + i \int_{t-A}^{t} \int_{t_0}^{s} V_1P_c(H_1)e^{-iH_1(s-\tau)}V_2(-\tau\vec{e}_1)U(\tau,t_0)P_s(t_0)\,d\tau\,ds. \]

Consider the decomposition

\[ L_{3,c} = I + iK, \]

\[ I = \int_{t-A}^{t} e^{\frac{i}{2}\Delta(t-s)}V_1P_c(H_1)e^{-iH_1(s-t_0)}P_s(t_0)\,ds, \]

\[ K = \int_{t-A}^{t} e^{\frac{i}{2}\Delta(t-s)}V_1 \int_{t_0}^{s} P_c(H_1)e^{-iH_1(s-\tau)}V_2(-\tau\vec{e}_1)U(\tau,t_0)P_s(t_0)\,d\tau\,ds. \]

There is no problem with \( I \) by similar arguments for the free case with Lemma 3.5.

Next, we decompose \( K \) further as follows:

\[ K = J + S + Z, \]

\[ S = \int_{t-A}^{t} e^{\frac{i}{2}\Delta(t-s)}V_1 \int_{t_0}^{t_0+B} P_c(H_1)e^{-iH_1(s-\tau)}V_2(-\tau\vec{e}_1)U(\tau,t_0)P_s(t_0)\,d\tau\,ds, \]

\[ Z = \int_{t-A}^{t} e^{\frac{i}{2}\Delta(t-s)}V_1 \int_{t_0}^{s-B} P_c(H_1)e^{-iH_1(s-\tau)}V_2(-\tau\vec{e}_1)U(\tau,t_0)P_s(t_0)\,d\tau\,ds, \]

\[ J = \int_{t-A}^{t} \int_{s-B}^{s} e^{\frac{i}{2}\Delta(t-s)}V_1P_c(H_1)e^{-iH_1(s-\tau)}V_2(-\tau\vec{e}_1)U(\tau,t_0)P_s(t_0)\,d\tau\,ds. \]

For \( S \), a similar argument as for \( L_1 \) implies

\[ \| \langle x-x_0 \rangle^{-\sigma} S \langle x-x_1 \rangle^{-\sigma} \|_{2 \to 2} \]

\[ \lesssim C \int_{t-A}^{t} \frac{1}{(t-s)^\frac{3}{2}} \int_{t_0}^{t_0+B} \frac{1}{(s-\tau)^\frac{3}{2}} \| \langle x \rangle^{-\sigma} U(s,t_0)P_s(t_0)\langle x-x_1 \rangle^{-\sigma} \|_{2 \to 2} \,d\tau\,ds \]

\[ \lesssim C(B) \int_{t-A}^{t} \int_{t_0}^{t_0+B} \frac{1}{(t-s)^\frac{3}{2}} \frac{1}{(s-\tau)^\frac{3}{2}} \,ds\,d\tau \]

\[ \leq C(A,B) \frac{1}{(t-t_0)^2}. \]

As usual, the constant \( C \) does not depend on \( T \).

For the second piece, we also have

\[ \int_{0}^{T} \| \langle x-x(t) \rangle^{-\sigma} Su \|_{L_x^2}^2 \,dt \]

\[ \leq C \int_{0}^{T} \left( \int_{t-A}^{t} (s-t)^\frac{3}{2} \int_{t_0}^{t_0+B} (s-\tau)^{-\frac{3}{2}} \| \langle x \rangle^{-\sigma} U(s,t_0)P_s(t_0)u \|_{L_x^2} \,d\tau\,ds \right)^2 \,dt \]

\[ \lesssim C(B) A \| \langle x \rangle^{-\sigma} U(s,t_0)P_s u \|_{L_x^2((t_0,t_0+B), L_x^2)}^2 \]

\[ \lesssim C(A,B) \| u \|_{L^2}. \]
Next, for $Z$, following a similar argument to $L_2$, we obtain
\[
\left\| (x-x_0)^{-\sigma} Z(x-x_1)^{-\sigma} \right\|_{2 \to 2} \\
\lesssim \int_{t-A}^{t} \frac{1}{(t-s)^{\frac{3}{2}}} ds \int_{t_0+B}^{s-B} \frac{1}{(s-\tau)^{\frac{3}{2}}} \left\| (x)^{-\sigma} U(\tau, t_0) P_s(t_0)(x-x_1)^{-\sigma} \right\|_{2 \to 2} d\tau \\
\lesssim C(T) \int_{t-A}^{t} \int_{t_0+B}^{s-B} \frac{1}{(t-s)^{\frac{3}{2}}} \frac{1}{(s-\tau)^{\frac{3}{2}}} \frac{1}{(\tau-t_0)^{\frac{3}{2}}} d\tau ds.
\]

For the second estimate,
\[
\int_{0}^{T} \left\| (x-x(t))^{-\sigma} Zu \right\|_{L^2_s}^2 dt \\
\lesssim \int_{0}^{T} \left( \int_{t-A}^{t} (t-s)^{-\frac{3}{2}} \int_{t_0+B}^{s-B} (s-\tau)^{-\frac{3}{2}} \left\| (x)^{-\sigma} U(\tau, t_0) P_s(t_0)u \right\|_{L^2_s} d\tau ds \right)^2 dt \\
\lesssim h(B) A \left\| (x)^{-\sigma} U(\tau, t_0) P_s(t_0)u \right\|_{L^2_s((0,T), L^2_s)}^2 \\
\lesssim h(B) AC^2(T) \|u\|_{L^2_s}^2
\]
where as before,
\[h(B) \lesssim B^{-1}.\]
Therefore, when we pick $B$ large enough, we have satisfied all the conditions for the bootstrap argument.

Finally, we analyze $J$:
\[
J = \int_{t-A}^{t} \int_{s-B}^{s} e^{i\frac{1}{2} \Delta(t-s)} V_1 P_c(H_1) e^{-iH_1(s-\tau)} V_2(1-\tau \xi^1) U(\tau, t_0) P_s(t_0) d\tau ds.
\]
We decompose the integral into low and high frequency parts:
\[
J_L = \int_{t-A}^{t} \int_{s-B}^{s} e^{i\frac{1}{2} \Delta(t-s)} V_1 F(|\vec{\sigma}| \leq M) e^{-iH_1(s-\tau)} P_c(H_1) V_2(1-\tau \xi^1) U(\tau, t_0) P_s(t_0) d\tau ds,
\]
\[
J_H = \int_{t-A}^{t} \int_{s-B}^{s} e^{i\frac{1}{2} \Delta(t-s)} V_1 F(|\vec{\sigma}| \geq M) e^{-iH_1(s-\tau)} P_c(H_1) V_2(1-\tau \xi^1) U(\tau, t_0) P_s(t_0) d\tau ds,
\]
where $F(|\vec{\sigma}| \leq M)$ and $F(|\vec{\sigma}| \geq M)$ denote smooth projections onto frequencies $|\vec{\sigma}| \leq M$ and $|\vec{\sigma}| \geq M$ respectively.

To analyze the low frequency part, we observe that for arbitrary $\epsilon > 0$,
\[
\int_{s-B}^{s} (-\tau \xi^1)^{-\frac{3}{2}} d\tau \leq \epsilon
\]
provided $s$ is large enough.

Set $V_2^\sigma(x) = V_2(x) \langle x \rangle^\sigma$, then we look at the following quantity,
\[
\left\| V_1 F(|\vec{\sigma}| \leq M) e^{-i\frac{1}{2} (s-\tau) \Delta} V_2^\sigma (1-\tau \xi^1) u \right\|_{L^2} = \left\| \int_{\mathbb{R}^3} K(x, \eta) \tilde{u}(\eta) d\eta \right\|_{L^2}
\]
where
\[
K(x, \eta) = V_1(x) \int_{\mathbb{R}^3} e^{-i\frac{1}{2} (s-\tau) \xi^2 + i\xi(x+\tau \xi^1)} \chi \left( \frac{\xi}{M} \right) \tilde{V}_2^\sigma (\xi - \eta) e^{-i\eta \tau \xi^1} d\xi.
\]
Observe that

\[ |K(x, \eta)| \leq C_M \langle x \rangle^{-N} \langle \tau \hat{e}_1 \rangle^{-N} \langle \eta \rangle^{-N}. \]

This decay result follows from the following two facts:

Integration by parts with

\[ e^{iL_2^{\xi} \hat{e}_1} = \left( \frac{\tau \hat{e}_1 \nabla \xi}{|\tau \hat{e}_1|^2} \right)^N e^{iL_2^{\xi} \hat{e}_1}; \]

and the decay estimate:

\[ \left| D^\beta \hat{V}_2^\sigma (\xi - \eta) \right| \lesssim \langle \eta \rangle^{-N}, \ |\xi| \lesssim M. \]

So we can conclude that for any \( N > 0, \)

\[ \left\| V_1 F (|\tilde{p}| \leq M) e^{-i\frac{\tau}{2} (s-\tau) \Delta} V_2^\sigma (\cdot - \tau \hat{e}_1) \right\|_{2 \rightarrow 2} \leq C_{N,M} \langle \tau \hat{e}_1 \rangle^{-N}. \]

By some similar calculations in [5], we conclude

\[ \left\| V_1 F (|\tilde{p}| \leq M) e^{-i(s-\tau)H_1} P_e (H_1) V_2^\sigma (\cdot - \tau \hat{e}_1) \right\|_{2 \rightarrow 2} \leq C_{N,M} \langle \tau \hat{e}_1 \rangle^{-N}. \]

But in our particular situation, one can do easy calculations based on Duhamel formula,

\[ e^{-i(s-\tau)H_1} = e^{-i(s-\tau)H_0} - i \int_{\tau}^s e^{-i(s-\tau)H_0} V_1 e^{-i(r-\tau)H_1} \, dr \]

\[ = F (|\tilde{p}| \leq M) e^{-i(s-\tau)H_0} V_1 e^{-i(s-\tau)H_1} \]

\[ -i \int_{\tau}^s F (|\tilde{p}| \leq M) e^{-i(s-\tau)H_0} V_1 e^{-i(s-\tau)H_1} V_2^\sigma (\cdot - \tau \hat{e}_1) \, dr. \]

\[ = \int_{\tau}^s \left[ e^{-i(s-\tau)H_0} V_1 F \right] (|\tilde{p}| \leq M) e^{-i(s-\tau)H_1} V_2^\sigma (\cdot - \tau \hat{e}_1) \, dr \]

\[ + \int_{\tau}^s e^{-i(s-\tau)H_0} [V_1, F (|\tilde{p}| \leq M)] e^{-i(s-\tau)H_1} V_2^\sigma (\cdot - \tau \hat{e}_1) \, dr. \]

Notice from construction, \( 0 \leq s - \tau \leq B, \)

\[ \left\| \int_{\tau}^s e^{-i(s-\tau)H_0} [V_1, F (|\tilde{p}| \leq M)] e^{-i(s-\tau)H_1} V_2^\sigma (\cdot - \tau \hat{e}_1) \, dr \right\|_{L^2} \leq \frac{B}{M} \| V_2^\sigma \|_{L^2}. \]
By Gronwall’s inequality, with the fact $0 \leq s - \tau \leq B$, one has

$$
\int_{s-B}^s \left\| V_1 F (|\vec{p}| \leq M) e^{-i(s-\tau)H_1} P_c (H_1) V_2^\sigma (\cdot - \tau \vec{e}_1) \right\|_{2 \to 2} \lesssim e^B \int_{s-B}^s \left( C_{N,M} \left\langle \tau \vec{e}_1 \right\rangle^{-N} + \frac{B}{M} \left\| V_2^\sigma \right\|_{L^2} \right) d\tau
$$

(44)

$$
\lesssim e^B \int_{s-B}^s \left( C_{N,M} \left\langle \tau \vec{e}_1 \right\rangle^{-N} + \frac{B}{M} \left\| V_2^\sigma \right\|_{L^2} \right) d\tau
\lesssim \epsilon + \frac{B^2 e^B}{M} \lesssim \epsilon
$$

provided $M$ is large enough.

Therefore, for $J_L$,

$$
\left\| \langle x - x_0 \rangle^{-\sigma} J_L \langle x - x_1 \rangle^{-\sigma} \right\|_{2 \to 2} \leq CC(T) \int_{t-A}^t \int_{s-B}^s \langle t-s \rangle^{-\frac{3}{2}} \langle s-\tau \rangle^{-\frac{3}{2}} \left\langle \tau \vec{e}_1 \right\rangle^{-\frac{3}{2}} ds d\tau
\leq CC(T) \langle t-t_0 \rangle^{-\frac{3}{2}} \int_{t-A}^t \int_{s-B}^s \langle t-s \rangle^{-\frac{3}{2}} \left\langle \tau \vec{e}_1 \right\rangle^{-\frac{3}{2}} ds d\tau
\leq \epsilon CC(T) \langle t-t_0 \rangle^{-\frac{3}{2}} .
$$

So when $A, B$ is large, we conclude that the coefficient satisfies the bootstrap conditions.

For the second part,

$$
\left\| \langle x - x(t) \rangle^{-\sigma} J_L u_0 \right\|_{L^2((0,T),L^2)} \leq C \int_{t-A}^t ds \int_{s-B}^s \langle t-s \rangle^{-\frac{3}{2}} \left\langle -\tau \vec{e}_1 \right\rangle^{-\frac{3}{2}} \left\| \langle x - \tau \vec{e}_1 \rangle^{-\sigma} U(\tau,t_0) P_s(t_0) u_0 \right\| \leq CC(T) \sqrt{A} \left\| \left\langle -\tau \vec{e}_1 \right\rangle^{-\frac{3}{2}} \right\|_{L^2(s-B,s)} \left\| U(\tau,t_0) P_s(t_0) u_0 \right\|_{L^\infty((0,T),L^2)} \leq CC(T) \sqrt{A} \epsilon \left\| u_0 \right\|_{L^2} .
$$

Again, we know when $\epsilon$ is small, we recapture the bootstrap conditions.

Finally, we need to check $J_{c,H}$. We will use the following version of the Kato smoothing estimate, or we can apply a variant of Kato’s smoothing estimate from [10].

**Lemma 3.6 ([8]).** For $\sigma > \frac{1}{2}$, we have

$$
\int_{\mathbb{R}} \left\| \langle x \rangle^{-\sigma} \langle \vec{p} \rangle^{-\frac{1}{2}} \nabla e^{-i \frac{1}{2} (s-\tau) \Delta} \right\|_{2 \to 2}^2 d\tau \leq C \tau ,
$$

we also have

$$
\int_{\mathbb{R}} \left\| \langle x \rangle^{-\sigma} \langle \vec{p} \rangle^{-\frac{1}{2}} \nabla e^{-i \frac{1}{2} (s-\tau) H_1} P_c (H_1) \right\|_{2 \to 2}^2 d\tau \leq C \tau ,
$$

We will use Lemma 3.6, but for the sake of completeness, we formulate the result from [10].
Lemma ([10]). Let $H = -\frac{1}{2}\Delta + V$, $\|V\|_\infty < \infty$, set $\psi(t) = e^{-itH}\psi_0$, then for all $T > 1$ and $\alpha > 0$, we have

$$\sup_{x_0 \in \mathbb{R}^n} \int_0^T \int_{\mathbb{R}^n} \left| \nabla \left( \nabla^{-\frac{\alpha}{2}} \psi(x,t) \right) \right|^2 dx dt \leq C_{\alpha,n}T \left( 1 + \|V\|_\infty \right) \|\psi_0\|_{L^2}.$$  

Consider

$$\int_{s-B}^s \left\| V_1 F (\langle p \rangle \geq M) e^{-iH_1(s-t)} P_c (H_1) V_2 (\cdot - \tau \epsilon_1^*) \right\|_{2 \to 2} d\tau \leq \int_{s-B}^s \left\| V_1 F (\langle p \rangle \geq M) e^{-iH_1(s-t)} P_c (H_1) V_2 (\cdot - \tau \epsilon_1^*) \right\|_{2 \to 2} d\tau \leq B \frac{1}{M} \left( \int_{s-B}^s \left\| \langle \rho \rangle^{-\frac{1}{2}} \nabla V_1 e^{-iH_1(s-t)} P_c (H_1) V_2 (\cdot - \tau \epsilon_1^*) \right\|_{2 \to 2} d\tau \right)^{\frac{1}{2}}.$$

By Young’s inequality [10, 1],

$$\|V_1, F (\langle p \rangle \geq M)\|_{2 \to 2} \lesssim \frac{1}{M}.$$  

Also note that

$$\langle \rho \rangle^{-\frac{1}{2}} \nabla V_1 = V_1 \langle \rho \rangle^{-\frac{1}{2}} \nabla + \left[ \langle \rho \rangle^{-\frac{1}{2}} \nabla, V_1 \right],$$

and

$$\left\| \left[ \langle \rho \rangle^{-\frac{1}{2}} \nabla, V_1 \right] \right\|_{2 \to 2} \lesssim 1.$$  

So for the first estimate, with bootstrap assumption (40), we get

$$\|\langle x - x_0 \rangle^{-\sigma} J_H (x - x_1)^{-\sigma}\|_{2 \to 2} \leq C(T) (t - t_0)^{-\frac{3}{2}} \int_{t-A}^t (t - s)^{-\frac{3}{2}} \times \int_{s-B}^s \left\| V_1 F (\langle p \rangle \geq M) e^{-iH_1(s-t)} P_c (H_1) V_2 (\cdot - \tau \epsilon_1^*) \right\|_{2 \to 2} d\tau ds \lesssim C(T) (t - t_0)^{-\frac{3}{2}} \int_{t-A}^t (t - s)^{-\frac{3}{2}} \left( M^{-1}B + M^{-\frac{1}{2}}\sqrt{B}(1 + \sqrt{B}) \right) \lesssim C(T)ABM^{-\frac{1}{2}} (t - t_0)^{-\frac{3}{2}}.$$  

For the other estimate,

$$\|\langle x - x(t) \rangle^{-\sigma} J_H u_0\|_{L^2(0,T),L^2} \lesssim \left\| \int_{t-A}^t (t - s)^{-\frac{3}{2}} \int_{s-B}^s \left( M^{-1} + M^{-\frac{1}{2}}(B + \sqrt{B}) \right) \|\langle x - \tau \epsilon_1^* \rangle^{-\sigma} U(\tau, t_0) P_s(t_0)u_0\|_{L^2(0,T)} \right\|_{L^2(0,T)} \lesssim C(T)BM^{-\frac{1}{2}} \|u_0\|_{L^2}.$$  

So we can pick $M$ large, then the coefficient satisfies the bootstrap condition again.

To sum up, when we pick $A$, $B$ and $M$ large enough independent of $T$, if we have for $t \in [t_0, T]$

$$\|\langle x - x_0 \rangle^{-\sigma} U(t, t_0) P_s(x - x_1)^{-\sigma}\|_{2 \to 2} \leq C(T) \frac{1}{(t - t_0)^{\frac{3}{2}}}.$$
we can improve it to
\[
\left\| \langle x-x_0 \rangle^{-\sigma} U(t,t_0) P_s \langle x-x_1 \rangle^{-\sigma} \right\|_{2 \to 2} \leq \frac{1}{2} C(T) \frac{1}{(t-t_0)^{\frac{\sigma}{2}}} + C \frac{1}{(t-t_0)^{\frac{\sigma}{2}}}
\]
Therefore, we can make for \( t \in [t_0,T] \),
\[
\left\| \langle x-x_0 \rangle^{-\sigma} U(t,t_0) P_s \langle x-x_1 \rangle^{-\sigma} \right\|_{2 \to 2} \leq C,
\]
for some constant independent of \( T \). So we conclude
\[
\left\| \langle x-x_0 \rangle^{-\sigma} U(t,t_0) P_s \langle x-x_1 \rangle^{-\sigma} \right\|_{2 \to 2} \leq C
\]
holds for arbitrary \( t \) which shows Lemma 3.1.

For the second part we proceed analogously. Indeed, if we suppose
\[
\int_0^T \left\| \langle x-x(t) \rangle^{-\sigma} U(t,t_0) P_s u \right\|_{L^2}^2 \, dt \leq C \left\| u \right\|_{L^2}^2 + \frac{1}{2} C^2(T) \left\| u \right\|_{L^2}^2.
\]
then we can improve the estimate to
\[
\int_0^T \left\| \langle x-x(t) \rangle^{-\sigma} U(t,t_0) P_s u \right\|_{L^2}^2 \, dt \leq C \left\| u \right\|_{L^2}^2 + \frac{1}{2} C^2(T) \left\| u \right\|_{L^2}^2.
\]
So we can obtain a bound for
\[
\int_0^T \left\| \langle x-x(t) \rangle^{-\sigma} U(t,t_0) P_s u \right\|_{L^2}^2 \, dt \leq C \left\| u \right\|_{L^2}^2
\]
which is independent of \( T \). Therefore, we can send \( T \) to \( \infty \) above. Finally, we obtain
\[
\int_0^\infty \left\| \langle x-x(t) \rangle^{-\sigma} U(t,t_0) P_s u \right\|_{L^2}^2 \, dt \leq C \left\| u \right\|_{L^2}^2
\]
which establishes Lemma 3.2.

Remark 3.7. With Theorem 2.3, one can show Lemma 3.1 easily as the free case. Set \( s = t-t_0 \), first, if \( |s| \leq 1 \), clearly by \( \| U(t,t_0) \|_{2 \to 2} \leq 1 \) and the integrability condition in \( \mathbb{R}^3 \), i.e. \( \sigma > \frac{3}{2} \), we can get the desired result. If \( |s| \geq 1 \), we apply the dispersive estimate for the free motion, by Young’s inequality we get
\[
\left\| \langle x-x_0 \rangle^{-\sigma} U(t,t_0) P_s(t_0) \langle x-x_1 \rangle^{-\sigma} \right\|_{2 \to 2} \lesssim \left\| \langle x \rangle^{-\sigma} \right\|_{L^2}^2 \| U(t,t_0) P_s(t_0) \|_{1 \to \infty},
\]
and from Theorem 2.3,
\[
\| U(t,t_0) P_s(t_0) \|_{1 \to \infty} \lesssim |t-t_0|^{-\frac{3}{2}}.
\]
But we proved Lemma 3.1 together with Lemma 3.2, since the dispersive estimate might not be available in other contexts.

4. BOUNDEDNESS OF THE ENERGY

In this section, we use Strichartz estimates to show that the energy of the whole evolution of the charge transfer model is bounded independently of time. The asymptotic completeness of the Hamiltonian shown in [10] will be used. We will still consider the model with two potentials as in the previous section.

Proof of Theorem 1.4. From Theorem 2.4, we can write the evolution as: for some \( \phi_0 \in L^2(\mathbb{R}^3) \),
\[
U(t,0)\psi_0 = \sum_{r=1}^m A_r e^{-i\lambda_r t} u_r + \sum_{s=1}^\ell B_s e^{-i\mu s t} \mathcal{S}_{-\infty}^1(t) w_s + e^{-it\Delta} \phi_0 + R(t)
\]
where $g$ is the Galilei transformation. It is trivial to see the part associated with bound states and moving bound states,

$$
(46) \quad \sum_{r=1}^{m} A_r e^{-i\lambda_r t} w_r + \sum_{s=1}^{\ell} B_s e^{-i\mu_s t} g_{-t\vec{e}_1}(t) u_s
$$

has bounded energy. Indeed, to be more precise, we have

$$
\left\| \sum_{r=1}^{m} A_r e^{-i\lambda_r t} w_r + \sum_{s=1}^{\ell} B_s e^{-i\mu_s t} g_{-t\vec{e}_1}(t) u_s \right\|_{H^1} \lesssim \sum_{r=1}^{m} \| w_r \|_{H^1} \| \psi_0 \|_{L^2} + \sum_{s=1}^{\ell} \| u_s \|_{H^1} \| \psi_0 \|_{L^2}.
$$

So it suffices to consider

$$
(47) \quad \psi(t) := U(t,0)\psi_0 = e^{-it\frac{\Delta}{2}} \phi_0 + R(t)
$$

where

$$
\| R(t) \|_{L^2} \to 0, \ t \to \pm\infty.
$$

In other words, we might assume

$$
P_s(t)\psi(t) = \psi(t).
$$

Rewrite the equation,

$$
(48) \quad i\psi_t + \frac{\Delta}{2} \psi = V_1 \psi + V_2(\cdot - t\vec{e}_1)\psi.
$$

We can differentiate the equation (48) and set

$$
v = \partial_x \psi =: \tilde{\psi},
$$

then $\psi$ satisfies

$$
(49) \quad iv_t + \Delta v - V_1 v - V_2(\cdot - t\vec{e}_1)v = \partial_1 V_1 \psi + \partial_1 V_2(\cdot - t\vec{e}_1)\psi.
$$

Again, it suffices to consider $\psi$ is in the scattering space. Since other components are easily to be bounded. To see this, we look at

$$
\langle v, w_r \rangle_{L^2} = -\left\langle e^{-it\frac{\Delta}{2}} \phi_0 + R(t), \partial_x w_r \right\rangle_{L^2}.
$$

By the asymptotic completeness result, we know

$$
\| R(t) \|_{L^2} \to 0, \ t \to \pm\infty.
$$

In particular, we know

$$
\| R(t) \|_{L^2} \lesssim C,
$$

so

$$
\| \langle R(t), \partial_x w_r \rangle_{L^2} \|_{L^2} \lesssim \| R(0) \|_{L^2} \lesssim \| \psi_0 \|_{L^2}
$$

since from Agmon’s estimate, $\partial_x w_r$ is still exponentially decaying. Notice that

$$
\left\| e^{-it\frac{\Delta}{2}} \phi_0, \partial_x w_r \right\|_{L^2} \to 0, \ t \to \infty,
$$

since we can approximate $\phi_0$ by $\tilde{\phi}_n \in L^2 \cap L^1$ in $L^2$ and then by the dispersive estimate for the free equation

$$
\| e^{-it\frac{\Delta}{2}} \tilde{\phi}_n \|_{L^\infty} \lesssim \frac{1}{|t|^\frac{3}{2}} \| \tilde{\phi}_n \|_{L^1}.
$$

A similar discussion holds for $u_s$, we can conclude that

$$
\| P_b(H_1) v(t) \|_{L^2} + \| P_b(H_2, t) v(t) \|_{L^2} \to 0, \ t \to \pm\infty,
$$

where

$$
P_b(H^1) = \sum_{r=1}^{m} A_r e^{-i\lambda_r t} w_r + \sum_{s=1}^{\ell} B_s e^{-i\mu_s t} g_{-t\vec{e}_1}(t) u_s.
$$
We write
\[ \| P_b (H_1) v(t) \|_{L^2} + \| P_b (H_2, t) v(t) \|_{L^2} \lesssim \| \psi_0 \|_{L^2}. \]

By the above argument, we can actually conclude that \( v \) is asymptotically orthogonal to the bound states of \( H_1 \) and moving bound states associated to \( H_2(t) \). We can in fact obtain an explicit rate of decay for the term
\[ \| P_b (H_1) v(t) \|_{L^2} + \| P_b (H_2, t) v(t) \|_{L^2} \]
go to 0, but it is enough for our purposes to know that it is just bounded by \( \| \psi_0 \|_{H^1} \).

Then by Proposition 2.5,
\[ \| (1 - P_s(t)) v(t) \|_{L^2} \lesssim \| \psi_0 \|_{H^1}. \]

Therefore, it is sufficient to estimate
\[ \| P_s(t) v(t) \|_{L^2} \]
and hence, without loss of generality, we assume
\[ P_s(t) v(t). \]

We do a similar argument as the proof for Strichartz estimates, Theorem 1.3.

Setting \( F(x, t) = \partial_t V_1 \psi + \partial_t V_2 (\cdot - te_1) \psi \), we can write (49) in the form
\[ i\psi_t + \frac{\Delta}{2} \psi = V_1 \psi + V_2 (\cdot - te_1) \psi + F(x, t). \]

By the endpoint Strichartz estimate for the free Schrödinger equation, we obtain
\[ \| v \|_{L^6_t L^6_x} \leq C \| V_1 v + V_2 (\cdot - te_1) v + F \|_{L^6_t L^6_x} \]
\[ \| V_1 v + V_2 (\cdot - te_1) v + F \|_{L^6_t L^6_x} \leq \| V_1 v \|_{L^6_t L^6_x} + \| V_2 (\cdot - te_1) v \|_{L^6_t L^6_x} + \| F \|_{L^6_t L^6_x}, \]
\[ \| V_1 v \|_{L^6_t L^6_x} \leq C \| \langle x \rangle^{-m} v \|_{L^6_t L^6_x}, \]
\[ \| V_2 (\cdot - te_1) v \|_{L^6_t L^6_x} \leq C \| \langle x - te_1 \rangle^{-m} v \|_{L^6_t L^6_x}. \]

So it suffices to estimate
\[ \| \langle x - x(t) \rangle^{-m} v \|_{L^6_t L^6_x} \]
for \( x(t) \) a smooth curve in \( \mathbb{R}^3 \).

By Duhamel’s formula,
\[ v(t) = e^{i \frac{\Delta}{2} t} v_0 + i \int_0^t e^{i \frac{\Delta}{2} (t-s)} (V_1 + V_2 (\cdot - se_1)) v(s) \, ds + i \int_0^t e^{i \frac{\Delta}{2} (t-s)} F(x, s) \, ds. \]

We write
\[ v(t) = U_1 + iU_2 + iU_3. \]

Certainly, it is easy to bound \( U_1 \) as Lemmas 3.3 and 3.4.

Next we bound \( U_3 \). We again apply Hölder’s inequality and the endpoint Strichartz estimate,
\[ \| \langle x - x(t) \rangle^{-m} \int_{t_0}^t e^{i \frac{\Delta}{2} (t-s)} F(x, s) \, ds \|_{L^6_t L^6_x} \leq \| \int_{t_0}^t e^{i \frac{\Delta}{2} (t-s)} F(x, s) \, ds \|_{L^6_t L^6_x} \leq \| F \|_{L^6_t L^6_x}. \]
It remains to bound
\[ \left\| \int_0^t e^{it\frac{1}{2} \Delta (t-s)} (V_1 + V_2(-se_1^1)) v(s) \, ds \right\|_{L_t^2 L_x^2}. \]

Rewrite
\[ v(s) = U(s,0)v_0 + i \int_0^s U(s,\tau)F(x,\tau) \, d\tau \]

By our assumption:
\[ \|v\|_{L_t^\infty L_x^2} \leq \|U\|_{L_t^2 L_x^2} \|v_0\|_{L_x^2} + \|\langle x \rangle^{-\beta} P_x U(t,0)v_0\|_{L_t^2 L_x^2} \]

By Lemma 3.1, Lemma 3.2, we have
\[ \left\| \langle x - x(t) \rangle^{-m} \int_0^t e^{it\frac{1}{2} \Delta (t-s)} (V_1 + V_2(-se_1^1)) P_x U(s,0)v_0 \, ds \right\|_{L_t^\infty L_x^2} \]

Also, we can get
\[ \left\| \langle x - x(t) \rangle^{-m} \int_0^t e^{it\frac{1}{2} \Delta (t-s)} (V_1 + V_2(-se_1^1)) \int_0^s P_x U(s,\tau)F(x,\tau) \, d\tau ds \right\|_{L_t^\infty L_x^2} \]

So we have shown that
\[ \|v\|_{L_t^\infty L_x^2} \leq C \|V_1 v + V_2(-te_1^1)v + F\|_{L_t^\infty L_x^{\frac{q}{2}}} \leq \|v_0\|_{L_x^2} + \|\langle x \rangle^\alpha F\|_{L_t^\infty L_x^2} + \|F\|_{L_t^\infty L_x^{\frac{q}{2}}} \]

for any Schrödinger admissible pair \((p,q)\).

Plugging in \(F = \partial_t V_1 \psi + \partial_t V_2(-te_1^1)\psi\), it is easy to estimate
\[ \|\langle x \rangle^\alpha F\|_{L_t^\infty L_x^2} + \|F\|_{L_t^\infty L_x^{\frac{q}{2}}} \lesssim \|\psi_0\|_{L^2}. \]

For the second piece, we use
\[ \|\partial_t V_1 \psi + \partial_t V_2(-te_1^1)\psi\|_{L_t^\infty L_x^{\frac{q}{2}}} \leq \|\partial_t V_1 \psi\|_{L_t^\infty L_x^{\frac{q}{2}}} + \|\partial_t V_2(-te_1^1)\psi\|_{L_t^\infty L_x^{\frac{q}{2}}} \]

By Hölder’s inequality, we have
\[ \|\langle x \rangle^\alpha (\partial_t V_1 \psi + \partial_t V_2(-te_1^1)\psi)\|_{L_x^2} \lesssim \|\psi\|_{L_x^2}. \]

Then applying the endpoint Strichartz estimate to \(\psi\) by Theorem 1.3, we get
\[ \|\langle x \rangle^\alpha (\partial_t V_1 \psi + \partial_t V_2(-te_1^1)\psi)\|_{L_t^\infty L_x^2} \lesssim \|\psi\|_{L_t^\infty L_x^2} \lesssim \|\psi\|_{L_x^2}. \]
So in particular, we infer that
\[ \|v\|_{L^\infty_t L^2_x} \lesssim \|v_0\|_{L^2} + \|\psi_0\|_{L^2} = \|\psi_0\|_{H^1}. \]
The same argument applies to all other partial derivatives of \( \psi \). So we can conclude that
\[ \sup_{t \in \mathbb{R}} \|\psi(t)\|_{H^1} \lesssim \|\psi_0\|_{H^1}. \]
The theorem is proved \( \Box \)

By a simple inductive argument, we obtain the following corollary:

**Corollary 4.1.** For \( \psi_0 \in H^k(\mathbb{R}^3) \) where \( k \) is a non-negative integer, then
\[ \sup_{t \in \mathbb{R}} \|U(t,0)\psi_0\|_{H^k} \leq C \|\psi_0\|_{H^k}. \]

**Remark.** As a concluding remark, we notice that we proved the boundedness of the energy based on Strichartz estimates and the asymptotic completeness of the Hamiltonian. In [5], Graf proved the asymptotic completeness based on the boundedness of the energy. So, we can see, modulo some technical assumptions on the spectrum of the Schrödinger operator, the boundedness of the energy is equivalent to the asymptotic completeness of the Hamiltonian. Also note that the asymptotic completeness can be also proved by the dispersive estimate as in [10].

5. **Matrix charge transfer models**

In this section, we extend our above results to matrix charge transfer models in \( \mathbb{R}^3 \) similarly as the work in [10]. For the sake of completeness, we start from the basic definitions following [10].

**Definition 5.1.** By a matrix charge transfer model we mean a system
\[
\frac{1}{i} \partial_t \vec{\psi} + \begin{pmatrix} -\frac{1}{2} \Delta & 0 \\ 0 & \frac{1}{2} \Delta \end{pmatrix} \vec{\psi} + \sum_{j=1}^{n} V_j (\cdot - \vec{v}_j t) \vec{\psi} = 0, \quad \vec{\psi}|_{t=0} = \vec{\psi}_0
\]
where \( \vec{v}_j \) are distinct vectors in \( \mathbb{R}^3 \), and \( V_j \) are matrix potentials of the form
\[
V_j(t, x) = \begin{pmatrix} U_j(x) & -e^{i\theta_j(t,x)}W_j(x) \\ e^{-i\theta_j(t,x)}W_j(x) & -U_j(x) \end{pmatrix},
\]
where \( \theta_j(t, x) = (|v_j|^2 + \alpha_j^2)t + 2x \cdot \vec{v}_j + \gamma_j \) with \( \alpha_j, \gamma_j \in \mathbb{R} \) and \( \alpha_j \neq 0 \). Furthermore, we require that each
\[
H_j = \begin{pmatrix} -\frac{1}{2} \Delta + \frac{1}{2} \alpha_j^2 & U_j \\ W_j & -\frac{1}{2} \Delta - \frac{1}{2} \alpha_j^2 - U_j \end{pmatrix}
\]
satisfies the admissible conditions (Definition 5.2) and stability condition (Definition 5.3) defined below.

Here we give the definitions of stability condition and admissible conditions for a matrix Hamiltonian \( A = B + V \) where
\[
B = \begin{pmatrix} -\frac{1}{2} \Delta + \mu & 0 \\ 0 & \frac{1}{2} \Delta - \mu \end{pmatrix}, \quad V = \begin{pmatrix} U & -W \\ W & -U \end{pmatrix}
\]
with \( \mu > 0 \) and \( U, W \) are of real-valued.
Definition 5.2. Let $A = B + V$ as above with $V$ exponentially decaying. We call the operator $A$ on $\mathcal{H} := L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ admissible provided the following hold:

1. $\text{spec}(A) \subset \mathbb{R}$ and $\text{spec}(A) \cap (-\mu, \mu) = \{\omega_\ell : 0 \leq \ell \leq M\}$, where $\omega_0 = 0$ and all $\omega_j$ are distinct eigenvalues. There are no eigenvalues in $\text{spec}_{\text{ess}}(A)$.
2. For $1 \leq \ell \leq M$, $L_\ell := \ker(A - \omega_\ell)^2 = \ker(A - \omega_\ell)$ and $\ker(A) \subset \ker(A^2) = \ker(A^3) =: L_0$. Moreover, these spaces are finite-dimensional.
3. The ranges $\text{Ran}(A - \omega_\ell)$, for $1 \leq \ell \leq M$ and $\text{Ran}(A^2)$ are closed.
4. The spaces $L_\ell$ are spanned by exponentially decreasing functions in $\mathcal{H}$ (say, with bound $e^{-c_0|x|}$).
5. All these assumptions hold as well for the adjoint $A^*$. We denote the corresponding (generalized) eigenspaces by $L^*_\ell$.
6. The points $\pm \mu$ are not resonances of $A$.

Remark. For detailed definition of resonance here, one can find it in [10] Remark 7.10.

Following the above admissible conditions for $A$, we have can define analogous projections onto continuous spectrum and point spectrum following [10] Lemma 7.3.

Lemma ([10], Lemma 7.3). There a direct sum decomposition

$$\mathcal{H} = \sum_{j=1}^M L_j + \left(\sum_{j=1}^M L_j^*\right)^\perp.$$  

The decomposition is invariant under $A$. Let $P_c$ denote the projection onto $(\sum_{j=1}^M L_j^*)^\perp$ and set $P_b = \text{Id} - P_c$. Notice that here $P_c$ is not an orthogonal projection. It is easy to see $AP_c = P_cA$, and there exist numbers $c_{ij}$ such that

$$P_b = \sum_{i,j} \phi_j c_{ij} \langle f, \psi_i \rangle, \quad \forall f \in \mathcal{H}$$

where $\phi_j$ and $\psi_i$ are exponentially decreasing functions.

Definition 5.3. For $A$ satisfying the admissible conditions, we say $A$ satisfies the stability condition if

$$\sup_{t \in \mathbb{R}} \|e^{itA}P_c\|_{\mathcal{H} \to \mathcal{H}} < \infty.$$  

In order the study the matrix charge transfer model, we need the vector-valued Galilei transformation similarly as in the scalar case:

$$G_{\vec{v},y}(t) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} := \begin{pmatrix} \mathbf{g}_{\vec{v},y}(t) \psi_1 \\ \mathbf{g}_{\vec{v},y}(t) \psi_2 \end{pmatrix},$$

where $\mathbf{g}_{\vec{v},y}(t)$ is the scalar version Galilei transformation. In contrast to the scalar case, the conjugated transformation now involves a modulation $M(t)$. We cite Lemma 8.2 in [10].

Lemma ([10], Lemma 8.2). Let $\alpha \in \mathbb{R}$ and let

$$A := \begin{pmatrix} -\frac{1}{2} \Delta + \frac{1}{2} \alpha^2 + U & -W \\ \frac{1}{2} \Delta - \frac{1}{2} \alpha^2 - U \end{pmatrix}.$$  


with real-valued $U$ and $W$. Moreover, let $\vec{v} \in \mathbb{R}^3$, $\theta(t, x) = (|\vec{v}|^2 + \alpha^2) t + 2x \cdot \vec{v} + \gamma, \gamma \in \mathbb{R}$, and define

$$H(t) := \begin{pmatrix} \frac{1}{2} \Delta + U (\cdot - \vec{v} t) & -e^{i\theta(t, \cdot - \vec{v} t)} W (\cdot - \vec{v} t) \\ e^{-i\theta(t, \cdot - \vec{v} t)} W (\cdot - \vec{v} t) & \frac{1}{2} \Delta - \frac{1}{2} \alpha^2 - U (\cdot - \vec{v} t) \end{pmatrix}.$$ 

Let $S(0) = \text{Id}$, $S(t)$ denote the propagator of the system

$$\frac{1}{i} \partial_t S(t) + H(t) S(t) = 0.$$

Finally, let

$$M(t) = M_{\alpha, \gamma}(t) = \begin{pmatrix} e^{-2\omega(t)} & 0 \\ 0 & e^{i\omega(t)} \end{pmatrix}$$

where $\omega(t) = \alpha^2 t + \gamma$. Then we have the following relation

$$S(t) = \mathcal{G}_{\vec{v}}(t)^{-1} M(t)^{-1} e^{-itA} M(0) \mathcal{G}_{\vec{v}}(0).$$

For matrix charge transfer models, the analysis should be similar to the scalar case except that we have to modify the asymptotic orthogonality condition. Recall that as we remarked above, it is not necessary to use the asymptotic completeness results. In the scalar case, the asymptotic orthogonality condition is sufficient for us. In the matrix case, the asymptotic orthogonality condition is replaced by the definition of “scattering states” in Definition 8.3 in [10] which is similar to the scattering space in the sense of Definition 1.2 for the scalar case.

**Definition 5.4.** Let $U(t) \tilde{\psi}_0 = \tilde{\psi}(t, \cdot)$, we call that $\tilde{\psi}_0$ a scattering state relative to $H_j$ if

$$\left\| P_b (H_j, t) U(t) \tilde{\psi}_0 \right\|_{L^2} \to 0, \ t \to \infty.$$

Here

$$P_b (H_j, t) := \mathcal{G}_{\vec{v}_j}(t)^{-1} M_{\gamma_j}(t)^{-1} P_b (H_j) M_{\gamma_j}(t) \mathcal{G}_{\vec{v}_j}(t)$$

with $M_j(t) = M_{\alpha_j, \gamma_j}(t)$.

By the discussion in Section 8.3 in [10], if $\tilde{\psi}_0$ a scattering state relative to each $H_j$, we have the rate of convergence similar to the scalar case,

$$\left\| P_b (H_1, t) U(t) \tilde{\psi}_0 \right\|_{L^2} + \left\| P_b (H_2, t) U(t) \tilde{\psi}_0 \right\|_{L^2} \lesssim e^{-\alpha t} \left\| \tilde{\psi}_0 \right\|_{L^2}$$

for some $\alpha > 0$.

With all the preparations above, we now can formulate our Strichartz estimates for matrix charge transfer models.

**Theorem 5.5.** Consider the matrix charge transfer model as in Definition 5.1. We denote $\tilde{\psi}(t) = U(t, 0) \tilde{\psi}_0$ and assume $\tilde{\psi}_0$ is a scattering state relative to each $H_j$ in sense of Definition 5.4. Then for a Schrödinger admissible pair $(p, q)$ in $\mathbb{R}^3$, i.e.,

$$\frac{2}{p} + \frac{3}{q} = \frac{3}{2}$$

with $2 \leq q \leq \infty$, $p \geq 2$, we have

$$\left\| \tilde{\psi} \right\|_{L^q_p ([0, \infty), L^p_2)} \leq C \left\| \tilde{\psi}_0 \right\|_{L^2},$$

for some finite constant $C$. 

As in the scalar case, the proof Theorem 5.5 is based on certain weighted estimates which rely on a bootstrap argument. Since the proof is basically identical as with the scalar case, we do not carry out the details. We only discuss it briefly. Recall that in our proof, there are several important ingredients: dispersive estimates for stationary potentials, the boundedness of wave operators, the Kato smoothing estimate. All of them hold for the matrix case. For the dispersive estimates for stationary potentials, one can find details in [2, 10, 4]; for the boundedness of wave operators, the results are discussed in [2]; the Kato smoothing estimates can be obtained as for the scalar case in [10]. Hence with the remark at the beginning of the second section, and all the proofs above, we can conclude that Strichartz estimates hold for the matrix case.

**Remark.** With the dispersive estimate for matrix transfer models and the results on scattering states, we can follow the proof in [10] to prove the asymptotic completeness for matrix charge transfer Hamiltonians.

Similar to the scalar case, we also have the energy estimate.

**Theorem 5.6.** For \( \vec{\psi}_0 \in \mathcal{H}^1 := H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \), we have

\[
\sup_{t \in \mathbb{R}} \left\| U(t, 0) \vec{\psi}_0 \right\|_{\mathcal{H}^1} \leq C \| \psi_0 \|_{\mathcal{H}^1}.
\]

**Corollary 5.7.** For \( \vec{\psi}_0 \in \mathcal{H}^k := H^k(\mathbb{R}^3) \times H^k(\mathbb{R}^3) \) where \( k \) is a non-negative integer, then we have

\[
\sup_{t \in \mathbb{R}} \| U(t, 0) \psi_0 \|_{\mathcal{H}^k} \leq C \| \psi_0 \|_{\mathcal{H}^k}.
\]

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**References**


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